مدخل الى

ديناميكيات الموائع الحسابية

(CFD) (Computational Fluid Dynamics)

والحرق الحسابي

(**Numerical Combustion)**

including topic specific dictionnary english-arabic

Samir Mourad (Editor)
Translation English to Arab: …

منبني على:

(CFD) Introduction to Computational Fluid Dynamics

3rd edition

John F. Wendt (Editor), A von Karman Institute Book Authors of used part: J. Anderson, R. Grundmann

و

Theoretical and Numerical Combustion (Thierry Poinsot, Denis Veynante) and Introduction to Combustion – Concepts and Applications, 2nd edition (Stephen R. Turns)

و مراجع اخرى

هذا الاصدار ليس بكامل. آخر تعديل:

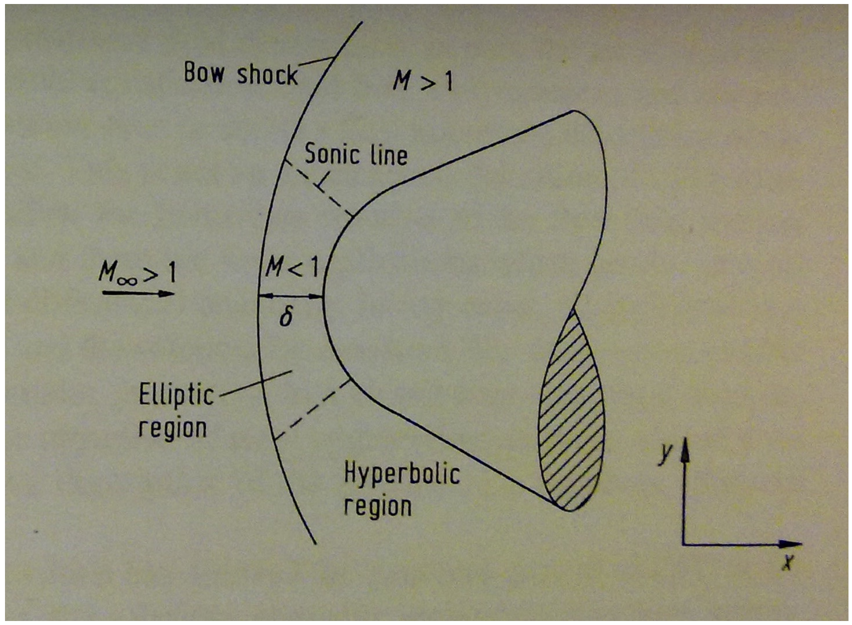
الاحد، 01 أيار، 2011

AECENAR
Association for Economical and Technological Cooperation in the Euro-Asian and North-African Region
www.aecenar.com

الفهرس

مدخل الى ديناميكيات الموائع الحسابية (Computational Fluid Dynamics) (CFD)

Samir Mourad (Editor)


Bow shock
M > 1
Sonic line
M∞ > 1
M < 1
δ
Elliptic region
Hyperbolic region
y
x

صورة 1.1

1.1 تعريفات اساسية[1]

ميكانيكا الموائع (Fluid Mechanics) هو تخصص فرعي من ميكانيكا المواد المتصلة (Mechanics Continuum) وهو معني أساسا **بالموائع،** التي هي أساسا السوائل والغازات، ويدرس هذا التخصص السلوك الفيزيائي الظاهر الكلي لهذه المواد، ويمكن تقسيمه من ناحية إلى إستاتيكا الموائع- أو دراستها في حالة عدم الحركة، أو ديناميكا الموائع أو دراستها في حالة الحركة، ويندرج تحتها تخصصات أخرى معينة، فهناك الديناميكيات الهوائية (**أيروديناميك**) والديناميكيات المائية (**هيدروديناميك**). يسعى هذا التخصص إلى تحديد الكميات الفيزيائية الخاصة بالموائع، وذلك مثل السرعة، الضغط، الكثافة، ودرجة الحرارة، **واللزوجة** ومعدل التدفق، وقد ظهرت تطبيقات

---

[1] من http://ar.wikipedia.org/wiki ولكن محقق من الكاتب

حسابية حديثة لإيجاد حلول للمسائل المتصلة بميكانيكا الموائع، ويسمى التخصص المعني بذلك ديناميكيات الموائع الحسابية (بالإنجليزية:Computational FluidDynamics) (CFD).

1.2 نظام الوحدات

النظام المستخدم هنا هو النظام العالمي للوحدات (SI).

القائمة أدناه تبين وحداته الاساسية:

الطول	الكتلة	الزمن	درجة الحرارة	القوة	الطاقة	القدرة	الضغط
m	kg	s	K	N	J	W	Pa
متر	كيلو غرام	ثانية	كلفن	نيوتن	جول	واط	باسكال

1.3 مضمون الجزء الاول من الكتاب

في الجزء الاول من هذا الكتيب يتناول ان شاء الله التالي:

- تلخيص لميكانيكا الموائع (بالإنجليزية: Fluid Mechanics)
- مدخل ملخص للتحليل عددي (بالإنجليزية: Numerics / Numerical Computation)

• اساليب ديناميكيات الموائع الحسابية (بالإنجليزية:Computational FluidDynamics)

يوجد بالغة العربية مرجع في المادة ميكانيكا الموائع و هو كتاب **ميكانيك الموائع** من محمد هاشم صديق[2].

1.4 الموائع (fluids)

الموائع كجمع لكلمة مائع (fluid) تشكل مجموعة من أطوار المادة، وهي أي مادة قابلة للانسياب تحت تأثير إجهاد القص وتأخذ شكل الإناء الحاوي لها. تتضمن الموائع كلاً من السوائل، الغازات، البلازما وأحيانا الأصلاب اللدنة plastic solids.

تصنف الموائع عادة إلى:

- **موائع قابلة للانضغاط (compressible fluids)** وهي الموائع التي تتغير كثافتها بتغير الضغط الواقع عليها مثل الغازات. و يسم ايضاً السريان الانضغاطي.
- **موائع غير قابلة للانضغاط (incompressible fluids)** وهي الموائع التي لا تتغير كثافتها بتغير الوضع الواقع عليها مثل السوائل. و يسم ايضاً السريان اللا انضغاطي.

---

[2] [Siddiq]

- موائع نيوتنية: **المائع النيوتني** هو مائع تكون فيه علاقة الإجهاد[3] – الانفعال (تشوه المواد نتيجة الإجهاد) علاقة خطية أي على شكل مستقيم يمر من مبدأ الإحداثيات، ويعرف اسم ثابت التناسب باللزوجة. سمي هذا المائع على اسم العالم اسحق نيوتن[4].

[3] engl. stress

[4] **إسحق "نيوتن"** (بالإنجليزية: Isaac Newton) وينادي **بالسير إسحق نيوتن** (4 يناير 1643– 31 مارس 1727) من رجال الجمعية الملكية كان فيزيائي إنجليزي وعالم رياضيات وعالم فلك وفيلسوف بعلم الطبيعة وكيمائي وعالم باللاهوت وواحدًا من أعظم الرجال تأثيرًا في تاريخ البشرية. ويعد كتابه *كتاب الأصول الرياضية للفلسفة الطبيعية* والذي نشر عام 1687 من أكثر الكتب تأثيرًا في تاريخ العلم واضعًا أساس لمعظم نظريات الميكانيكا الكلاسيكية. في هذا الكتاب، وصف "نيوتن" الجاذبية العامة وقوانين الحركة الثلاثة والتي سيطرت على النظرة العلمية إلى العالم المادي للقرون الثلاثة القادمة ووضح "نيوتن" أن حركة الأجسام على كوكب الأرض والتي لها أجرام سماوية تحكمها مجموعة القوانين الطبيعية نفسها عن طريق إثبات الاتساق بين قوانين "كبلر" الخاصة بالحركة الكوكبية ونظريته الخاصة بالجاذبية؛ ومن ثم إزالة الشكوك المتبقية التي ثارت حول نظرية مركزية الشمس مما أدى إلى تقديم الثورة العلمية. وفيما يتعلق بالميكانيكا، أعلن "نيوتن" مبادئ بقاء الطاقة الخاصة بكل من كمية الحركة وكمية الحركة الزاوية. وفي علم البصريات، اخترع "نيوتن" أول تلسكوب عاكس[3] عملي. وكذلك أيضًا طور نظرية الألوان (لون) معتمدًا على ملاحظة أن المنشور يحلل الضوء الأبيض إلى العديد من الألوان التي تشكل الطيف المرئي. وبالإضافة إلى ذلك، صاغ قانون نيوتن للتبريد ودرس سرعة الصوت. وبالنسبة لعلم الرياضيات، يشارك "نيوتن" "جوتفريد لايبنتز" في شرف تطوير حساب التكامل والتفاضل. وكذلك أيضًا، أثبت النظرية ذات الحدين المعممة وطور ما يسمى بـــ "طريقة نيوتن" الخاصة بتقريب الأصفار الموجودة بالدالة وساهم في دراسة متسلسلة القوى. تظل مكانة "نيوتن" الرفيعة بين العلماء في أعلى مرتبة الأمر الذي أثبته استطلاع رأي أجري عام 2005 فيما يتعلق بعلماء المجتمع الملكي البريطاني وكان السؤال الذي طرحه هذا الاستطلاع هو من كان له أعظم تأثير على تاريخ العلم "نيوتن" أم "ألبرت آينشتاين". وكانت نتيجة الاستطلاع هي أن "نيوتن" هو يعتبر الأكثر تأثيرًا.[4] علاوةً على ذلك، كان "نيوتن" تقيًا للغاية (على الرغم من أنه لم يكن متفقًا مع الأعراف الدينية القائمة) ومنتجًا للعديد من الأعمال في تفسيرات الكتاب المقدس أكثر مما أنتجه في العلوم الطبيعية التي لم ينس العالم إسهاماته به حتى الآن.

- موائع غير نيوتنية: **مائع لا نيوتوني** هو مائع لا يمكن وصف جريانه باستخدام ثابت اللزوجة. تعتبر أغلب المحاليل البولميرات والبوليمرات الذائبة من الموائع اللانيوتونية والكثير من السوائل الشائعة مثل الكتشب، ذائب النشا، الدم والشامبو.

1.5 الكمية المتصلة

يمكن اعتبار المائع كمية متصلة إذا كانت أصغر مسافة في التحليل أكبر من المتوسط المسار الحر للجزئيات.

$L >> 1$

1.6 الكثافة

باعتبار أن الحجم V_0 هو مكعب أصغر مسافة ترد عي التحليل وتستوفي شرط الكمية المتصلة فإن الكثافة ρ تعرف كما يلي:

$$\rho = \lim_{\Delta V \to V_0} \left(\frac{\Delta m}{\Delta V} \right)$$

حيث m الكتلة بالكيلوغرام و V الحجم بالمتر المكعب و وحدة الكثافة kg/m^3 .

1.7 الكثافة النسبية

هي كثافة المادة منسوبة إلى الكثافة المعيارية للماء ، وهي $1000 kg/m^3$.

$$s = \rho / \rho_w$$

1.8 قنون الغاز الكامل (ideal gas)

$$p = R\rho T \qquad (1.1)$$

يربط القانون الضغط المطلق للغاز p بالدرجة المطلقة للحرارة T و الكثافة ρ . R ثابت الغاز و قيمته للهواء $287 J\ K^{-1} kg^{-1}$.

1.9 السريان الرتيب (steady flow)

هو السريان الذي لا تتغير صفاته مع الزمن عند أي موضع محدد.

1.10 السريان المنتظم (uniform flow)

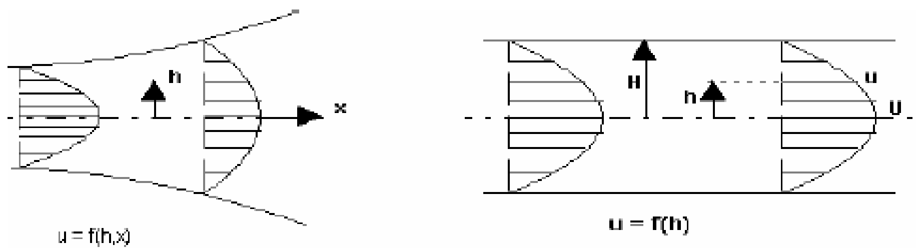
يوصف السريان بأنه منتظم عند مقطع إذا كانت قيمة كل من خواصه ثابتة في كل نقاط المقطع .

1.11 خط الانسياب (streamline)

يُعرف خط الانسياب بأنه الخط الذي تشكل المماسات له في كل أجزائه اتجاهات السرعة في وقت محدد.

1.12 أبعاد السريان (dimensions of flow)

يوصف السريان بأنه **أحادي، ثنائي أو ثلاثي البعد** بُناءً على العدد الأدنى من الإحداثيات المكانية التي يمكن أن يوصف بها.
الشكل (1.2) يعطي مثالاً لسريان أحادي البعد وآخر ثنائي البعد، حيث تعتمد السرعة على الإحداثي h في المثال الأول وتعتمد على الاحداثيين x و h في الثاني.



الشكل 1.2

1.13 الاجهاد (stress)

الإجهاد هو القوة السطحية العاملة على وحدة مساحة

$$\sigma = \lim_{\Delta A \to 0}\left(\frac{\Delta F}{\Delta A}\right)$$

و للإجهاد مركبتين إحداهما عمودية و الأخرى مماسة

$$\underline{\sigma} = \underline{\sigma}_n + \underline{\sigma}_t$$

ويُفضّل في ميكانيكا الموائع استخدام تعبير الضغط p في الاتجاه المتعامد حيث

$$\underline{\sigma}_n = -p\underline{n}$$

ويستخدم تعبير الإجهاد القصي τ في الاتجاه المماس حيث

$$\underline{\sigma}_t = \underline{\tau}$$

وبذلك

$$\underline{\sigma} = -p\underline{n} + \underline{\tau} \qquad (1.2)$$

1.14 السريان الصفائحي (laminar flow) السريان المائر (turbulent flow)

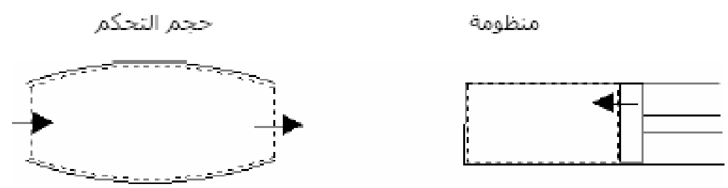
يتصف السريان الصفائحي بثبات الشكل والانسيابية بحيث يمكن اعتبار طبقاته تنزلق فوق بعضها البعض في شكل صفائح أو رقائق، بينما يتصف السريان المائر بالعنف والاضطراب.

يمكن استنباط الأسس التي تحكم تحول السريان من إحدى الحالتين إلى الأخرى بتأمل سريان الماء من صنبور. عند فتح الصنبور قليلاً نلاحظ انتظاماً في سريان الماء وثباتاً في شكله دون اضطراب كأنه مكون من صفائح أسطوانية تنزلق على بعضها البعض. يوصف هذا السريان بأنه صفائحي. بزيادة معدل السريان يمُور الماء و يضطرب ويفقد انتظامه ويوصف حينئذٍ بأنه مائر.

ويمكن إثبات أن التحول من الحالة الصفائحية إلى الحالة المائرة عند معدل سريان ثابت يحدث بزيادة السرعة أو زيادة القطر أو إنقاص اللزوجة. ويجمع المتغيرات الثلاثة مقدار لابُعدي يعرف بعدد رينولز Re يحكم التحول المذكور. و يحدث هذا التحول للسريان في الأنابيب في المدى $4000 \geq Re \geq 2000$. و يسمى عدد رينولز الذي يحدث عنده التحول **عدد رينولز الحرج** Re_c.

يتسم توزيع السرعة للسريان الصفائحي داخل الأنابيب بشكل المقطع المكافئ بينما يكون هذا التوزيع معقداً نسبياً في حالة السريان المائر.

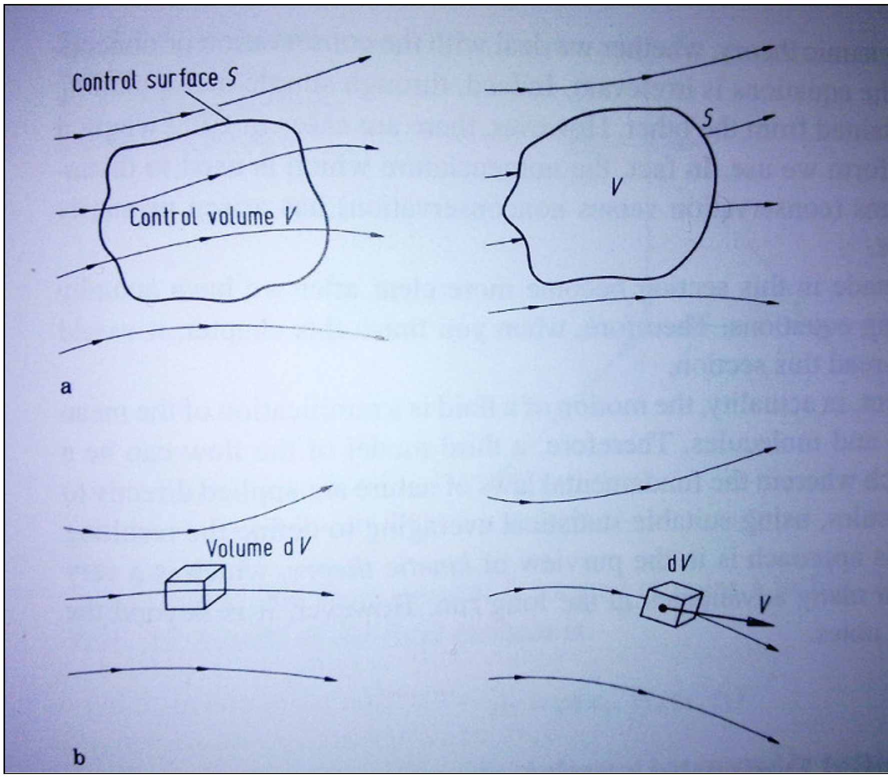
1.15 المنظومة (system) وحجم التحكم (control volume) و موحل في الصغر.عضو مائعي (infinitesimal fluid element)



الشكل 1.3

المنظومة معنية بكمية محددة من المادة يحدها عن بقية المائع جدار تخيلي أو حقيقي ويمكن أن يتغير موقعها وشكلها مع الوقت. حجم التحكم منطقة محددة وثابتة في المكان، ويمكن أن تتغير المادة داخل حجم التحكم مع الزمن. الشكل (1.3) يبين أمثلة للمفهومين.

هذا الحجم التحكم مرسوم في الشكل (1.3.1 a) على اليسار ولكن ايضاً يمكن ان ننظر الى حجم التحكم كما هو في الشكل (1.3.1 a) على اليمين و هو حجم التحكم يتحرك مع السريان.



الشكل (1.3.1)

([Wendt 2009], Fig. 2.1)

Fig. 1.3.1 a, left side: *finite control volume* V, an a finite control surface S *fixed in space*:
The fluid equations the we *directly* obtain by applying the fundamental physical principles to a finite control volume are in *integral form*.
These integral forms of the governing equations can be manipulated to *indirectly* obtain partial differential equations. The equations so obtained, in either integral or partial differential form, are called the *conservation form* of the governing equations.

The equations obtained from the finite control volume moving with the fluid (**Fig. 1.3.1 a, right side**), in either integral or partial differential form, are called the *non-conservation form* of the governing equations.

If we consider a infinitesimal fluid element, which is fixed is space (**Fig. 1.3.1 b, left side**), we can *directly* derive the partial differential equations. This is again the conservation form.
If we consider a infinitesimal fluid element, which is moving is space (**Fig. 1.3.1 b, right**

side), we can *directly* derive the partial differential equations. This is again the non-conservation form.	
In general aerodynamic theory, wheter we deal with the conservation or nonconservation forms of equations is irrelevant. However, there are cases in CFD where it is important which form we use.	

1.16 الضغط المقياسي والضغط الفراغي

الضغط المقياسي = الضغط المطلق - الضغط الجوي

الضغط الفراغي = - الضغط المقياسي

1.17 القوة الجسمية والقوة السطحية

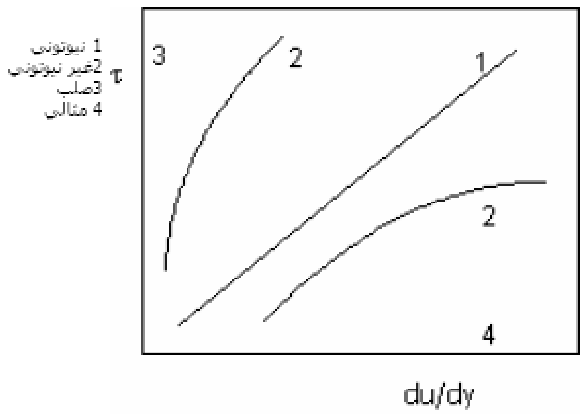
القوة الجسمية هي التي تنشأ عن كتلة الجسم مثل قوة الجاذبية والقوة الكهروماغنطيسية. والقوة السطحية هي تلك التي تعمل على سطح المادة وتنحصر في الضغط والقص.

1.18 الاجهاد القصي

تُنسب إلى نيوتن العلاقة النظرية بين الإجهاد القصي τ وممال السرعة في الاتجاه المتعامد $\frac{\partial u}{\partial y}$ للسريان الصفائحي وهي:

$$\tau = \mu \frac{\partial u}{\partial y} \qquad (1.3)$$

وقد أجريت تجارب للتحقق من المعادلة معملياً و عُلم أنها صحيحة لمعظم الموائع المستخدمة في التطبيقات الهندسية مثل الماء والهواء و الوقود النفطي. و سُمي ثابت المعادلة μ باللزوجة أو اللزوجة المطلقة أو اللزوجة الحركية، ووحدتها Pa.s . وتعرف الموائع التي تستجيب لهذه العلاقة عند درجة حرارة ثابتة بالموائع **النيوتونية** - الشكل (1.4).



الشكل (1.4)

تُسمى فصيلة الموائع التي لا تُعطي علاقة خطية بين القص ومعامل السرعة موائع **لانيوتونية**. أمثلةً لها البوية و النفط الشمعي.

تؤثر درجة الحرارة في قيمة اللزوجة حيث تنقص مع ازدياد الحرارة للسوائل وتزيد مع ازدياد الحرارة للغازات .

تُعرّف اللزوجة الكينماتية ν كما يلي:

$$\nu = \frac{\mu}{\rho}$$

ووحدتها m^2/s.

2 المعادلات الاساسية في ميكانيك الموائع (Governing Equations of Fluid Dynamics)

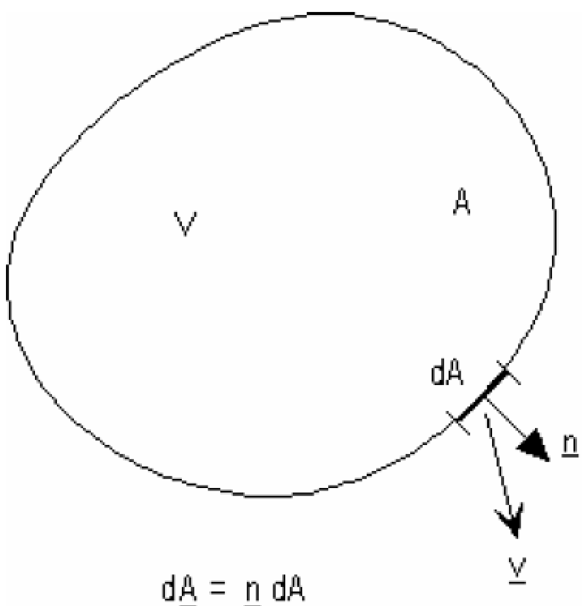
التالي منبني على [صديق]، فصل 2 و [Anderson 1991].

2.1 مدخل

الاساس في CFD هو المعادلاب الاساسية في ميكانيك الموائع و هي معادلات الحفظ الثلاث:
حفظ الكتلة(mass conservation) وحفظ الطاق (energy conservation) وحفظ كمية التحرك momentum (conservation). و قدم لذلك بتعريف متجه السريان الذي يشكل عنصراً مشتركاً في كل معادلات الحفظ.

2.1.1 متجه السريان

الشكل 2.1



الحجم التحكمي الموضح في الشكل (2.1) حجمه V و مساحته A. بالتركيز على المساحة التفاضلية dA فان الكتلة الخارجة عبرها هي dm في الوقت dt ليصبح معدل السريان $d\dot{m}$. سرعة السريان في الموضع هي المتجه **$\underline{v}$** تزاوية α مع المتجه أحادي الطول $\underline{n}$ المتعامد على المساحة dA حيث

$$d\underline{A} = \underline{n}\, dA$$

$$d\dot{m} = \rho\, d\dot{V} = \rho \boldsymbol{v} \cos\alpha\, dA = \rho\, \underline{v}.d\underline{A}$$

$\dot{m} =$ معدل سريان الكتلة عبر كل السطح

$$\dot{m} = \oiint_A \rho \underline{v}.d\underline{A} \quad \text{...}(2.1)$$

نُعرّف متجه سريان الكتلة كما يلي:

متجه سريان الكتلة = (متجه السرعة) (الكتلة في وحدة حجمية) = $\rho \underline{\boldsymbol{v}}$

وبالمثل:

متجه سريان الطاقة = (متجه السرعة) (الطاقة في وحدة حجمية)

$$=\rho \left(e + \frac{v^2}{2} + gz \right) \underline{\boldsymbol{v}}$$

وبالمثل:

متجه سريان كمية التحرك = (متجه السرعة) (كمية التحرك في وحدة حجمية)

$$= \rho\, u \underline{\boldsymbol{v}},\ \rho\, v\underline{\boldsymbol{v}},\ \rho\, w\underline{\boldsymbol{v}}$$

في الاتجاهات z, y, x على التوالي.

وبذلك فان معدل سريان الطاقة عبر السطح A =

$$\oiint \rho \left(e + \frac{v^2}{2} + gz \right) \underline{\boldsymbol{v}}.d\underline{A} \quad \text{..................................(2.2)}$$

ومعدل سريان كمية التحرك عبر السطح A =

$$\oiint_A \rho \underline{v} (\underline{v}.d\underline{A}) \quad \text{..(2.3)}$$

2.2 الاشتقاق الكبير (The Substantial Derivate)

كنموذج (model) للسريان سنأخذ الصورة التي هي على اليمين من الشكل 1.3.1(b) وهو

As a model for the flow, we will adopt the picture shown at the right of Fig. 1.3.1 (b).

Namely that of an **infinitesimally small fluid element moving with the flow**. The motion of the fluid element is shown in detail in Fig. 2.2.1.

Here, the fluid element is moving through cartesian space. The unit vectors along the x, y, z axis are $\vec{i}, \vec{j}, \vec{k}$.

The vector velocity field in this cartesian space is given by

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

Where the components of velocity are given respectively by

$$u = u(x, y, z, t)$$

$$v = v(x, y, z, t)$$

$$w = w(x, y, z, t)$$

Note that we are considering in general an *unsteady flow,* where u, v, and w are functions of both space and time, t. In addition the scalar density field is given by $\rho = \rho(x, y, z, t)$.

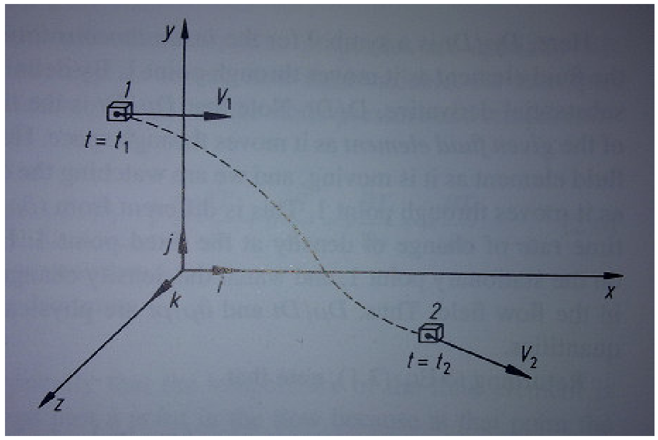


Fig. 2.2.1 ([Wendt 2009], Fig. 2.2)

At the time t_1 the fluid element is located at point 1 in Fig. 2.2.1. At this point and time, the density of the fluid element is $\rho_1 = \rho(x_1, y_1, z_1, t_1)$

At a later time t_2 the fluid element has moved to the point 2 where the density is $\rho_2 = \rho(x_2, y_2, z_2, t_2)$

Since $\rho = \rho(x, y, z, t)$, we can expand this function in a Taylor's series about point 1 as follows:

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{(higher order terms)}$$

With ignoring the higher order terms we obtain

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{z_2 - z_1}{t_2 - t_1}\right)\left(\frac{\partial \rho}{\partial z}\right)_1 + \left(\frac{\partial \rho}{\partial t}\right)_1 \quad (2.1.1)$$

Eq. (2.1.1) is physically the average time-rate-of-change in density of the fluid element as it moves from point 1 to point 2. In the limit, as t_2 approaches t_1, this term becomes

$$\lim_{t_2 \to t_1} \left(\frac{\rho_2 - \rho_1}{t_2 - t_1}\right) \equiv \frac{D\rho}{Dt}$$

$\frac{D\rho}{Dt}$ is a symbol for the *instantaneous* time rate of change of density.

By definition, this symbol is called the substantial derivate, D/Dt.

$\dfrac{D\rho}{Dt}$ is the time rate of change of density of the *given fluid element*. Our eyes are locked with the fluid element, not with the point in the space. So $\dfrac{D\rho}{Dt}$ is different physically and numerically from $\left(\dfrac{\partial \rho}{\partial t}\right)_1$ which is physically the time rate of change of density at the fixed point 1.

Returning to Eq. (2.1.1), note that

$$\lim_{t_2 \to t_1}\left(\frac{x_2 - x_1}{t_2 - t_1}\right) \equiv u$$

$$\lim_{t_2 \to t_1}\left(\frac{y_2 - y_1}{t_2 - t_1}\right) \equiv v$$

$$\lim_{t_2 \to t_1}\left(\frac{z_2 - z_1}{t_2 - t_1}\right) \equiv w$$

Thus, taking the limit of Eq.(2.1.1) as $t_2 - t_2$, we obtain

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \quad (2.1.2)$$

From (2.1.2) we obtain an expression for the substantial derivate in cartesian coordinates

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad (2.1.3)$$

In cartesian coordinates the vector operator ∇ is defined as

$$\nabla \equiv \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \quad (2.1.4)$$

Hence Eq.(2.1.3) can be written as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla) \quad (2.1.5)$$

Eq.(2.1.5) represents a definition of the substantial derivative operator in vector notation; thus it is valid for any coordinate system.

$\dfrac{\partial}{\partial t}$ is called the *local derivative* which is physically the time rate the time rate of change at a fixed point; $\vec{V} \cdot \nabla$ is called the *consecutive derivative,* which is physically the time rate of change due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different. The substantial derivative applies to any flow-field variable, for example, Dp/Dt, DT/Dt, …, where p and T are static pressure and temperature respectively.

The substantial derivative is essentially the same as the total differential from calculus. Therefore, the substantial dervative is nothing more than a total derivative with respect to time.

2.3 المعنى الفيزيائية من تباعد السرعة (**divergence of velocity**) $\nabla\cdot\vec{V}$

تباعد السرعة (divergence of velocity) $\nabla\cdot\vec{V}$

$\nabla\phi$

$$\nabla\cdot\vec{V}=\frac{1}{\delta V}\frac{D(\delta V)}{Dt} \quad (2.4)$$

$\nabla\vec{V}$ is physically the time rate of change of the volume of a moving fluid element, per unit volume.

$\nabla\vec{V}$ هو التغيير الزمني لحجم التحكم (control volume) من عضو مائع (fluid element) جارٍ (moving) و ذلك حسب الحجم التحكم (per control volume)

2.4 حفظ الكتلة (**mass conservation**)

صيغة قانون حفظ الكتلة مطبقاً على سريان المائع:

"معدل تراكم الكتلة داخل الحجم التحكمي مضافاً إليه خالص معدل سريان الكتلة إلى خارج الحجم التحكمي يساوي صفر.

الكتلة الكلية داخل الحجم التحكمي = $\oiiint_V \rho dV$

معدل ازدياد الكتلة داخل الحجم التحكمي (control volume):

$$\frac{\partial}{\partial t}\oiiint_V \rho dV=\oiiint_V \frac{\partial\rho}{\partial t}dV$$

لأن حدود التكامل لا تعتمد على الوقت.

من المعادلة (2.1) خالص سريان الكتلة إلى خارج الحجم التحكمي

$$=\oiint_A \rho \underline{v}.d\underline{A}$$

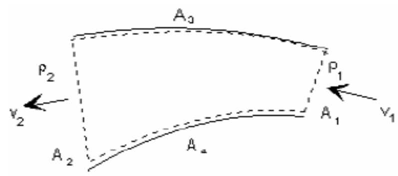
$$\oiiint_V \frac{\partial \rho}{\partial t} dV + \oiint_A \rho \underline{v}.d\underline{A} = 0 \quad ....(2.4)$$

المادلة (2.4) هي معادلة حفظ الكتلة في الصورة التكاملية (integral form).

تطبيق على سريان رتيب أحادي البعد (الشكل 2.2):

الحد الأول في المعادلة (2.4) يساوي صفر نسبةً لرتابة السريان.

السطحان (3) و (4) لا تعبرهما كتلة ولذلك يصير فيهما تكامل الحد الثانى من معادلة الكتلة صفراً .



الشكل 2.2

تُختزل معادلة الكتلة بذلك إلى الصورة :

$$\iint_{A1} \rho_1 \underline{v}_1 . d\underline{A}_1 + \iint_{A2} \rho_2 \underline{v}_2 . d\underline{A}_2 = 0$$

و بملاحظة أن المتجه $\underline{A}$ يتجه إلى خارج الحجم التحكمي

$$-\iint_{A1} \rho_1 v_1 dA_1 + \iint_{A2} \rho_2 v_2 dA_2 = 0$$

$$-\rho_1 \boldsymbol{v}_1 A_1 + \rho_2 \boldsymbol{v}_2 A_2 = 0$$

$\rho \boldsymbol{v} A =$ ثابت...(2.5)

2.4.1 معادلة الاستمرارية (continuity equation)

يطلق هذا الاسم عامةً على معادلة حفظ الكتلة في صورتها التفاضلية. بدءً من المعادلة (2.4) يمكن تحويل الحد الثاني من صورة التكامل السطحي الى صورة التكامل الحجمي باستخدام نظرية التباعد (divergence theorem).[5]

To obtain the basic equations of fluid motion,

---

[5]

1 اذا كانت $f = f(x, y, z)$
فان ممال f هو المتجه:

$$\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} \quad \ldots\ldots(1)$$

2 اذا كانت $\underline{\varphi}$ متجه ذا مركبات مطلقة φ_x و φ_y و φ_z في الاتجاهات x و y و z ، على التوالي ، فان التباعد لـ$\underline{\varphi}$

$$\nabla \cdot \underline{\varphi} = \frac{\partial \varphi_x}{\partial x} + \frac{\partial \varphi_y}{\partial x} + \frac{\partial \varphi_k}{\partial x} \quad \ldots\ldots(2)$$

3 تربط نظرية التباعد التكامل الحجمي و التكامل السطحي بالعلاقة

$$\iiint_V \left(\nabla \cdot \underline{\varphi}\right) dV = \iint_A \underline{\varphi} \cdot d\underline{A} \quad \ldots\ldots(3)$$

always the following way is followed:
Choose the appropriate fundamental physical principles from physics
Apply these physical principles to a suitable model of the flow.
From this application, extract the mathematical equations which embody such physical principles.
So, in our case the physical principle is:
"Mass is Conserved".

$$\oiiint_V \frac{\partial \rho}{\partial t} dV + \oiiint_V (\nabla . \rho \underline{v}) dV = 0$$

$$\oiiint_V \left(\frac{\partial \rho}{\partial t} + \nabla . \rho v \right) dV = 0$$

تبعاً لقوانين التكامل تكون قيمة المكامَل صفراً إذا كانت قيمة التكامل صفراً و كانت حدود التكامل اختياريةً.

$$\frac{\partial \rho}{\partial t} + \nabla . \rho \underline{v} = 0 \text{...(2.6a)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \text{.......................(2.6b)}$$

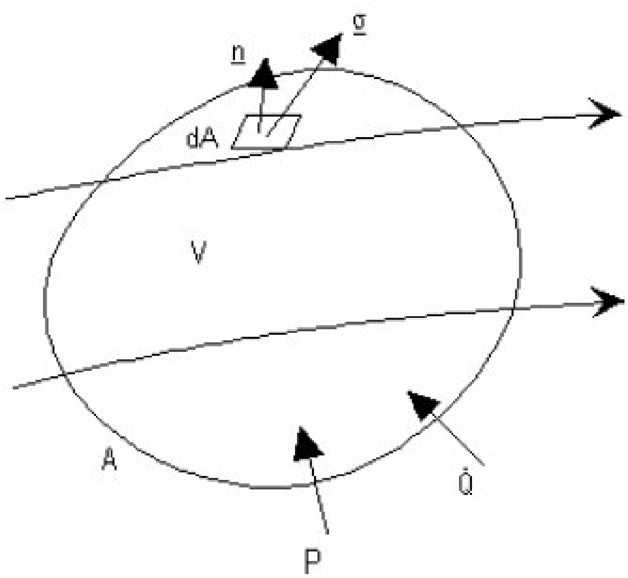
حيث w, v, u هي مركبات السرعة في الاتجاهات z , y , x .

و في حال ان السريان لا انضغاطي (incompressible flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{..............................(2.7)}$$

2.5 حفظ الطاقة (energy conservation)

الشكل 2.4



تُستمَد معادلة حفظ الطاقة من القانون الأول للحركيـة الحراريـة مطبقـاً علـى حجم تحكمي :

"معدل تراكم الطاقة داخل الحجم التحكمي مضافاً اليه خالص معدل سريان الطاقة إلى خارج الحجم التحكمي بانتقال الكتلة يعادل القدرة المبذولة على المائع داخل الحجم التحكمي مضافاً اليها معدل سريان الحرارة إلى داخل الحجم التحكمي" .

$$\frac{\partial}{\partial t}\oiiint_V \rho(e+\frac{v^2}{2}+gz)dV+\oiint_A \rho(e+\frac{v^2}{2}+gz)\underline{v}.d\underline{A}=\oiint_A(\underline{\sigma}.\underline{v})dA+P+\dot{Q}$$

الحدان الاوليان في جانب المعادلة الأيمن يعبران عن القدرة المبذولة على المائع داخل الحجم التحكمي، و $\dot{Q}$ معدل سريان الحرارة إلى داخل الحجم التحكمي.

بتجاهل اللزج (viscosity) يصبح الإجهاد (stress) σ:

$$\underline{\sigma}=-p\underline{n}$$

<-

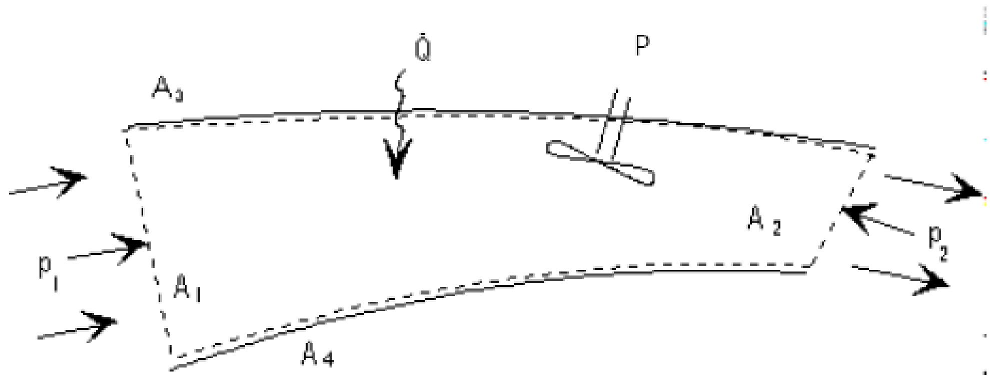
$$\oiiint_V \frac{\partial}{\partial t}[\rho(e+\frac{v^2}{2}+gz)]dV+\oiint_A \rho(e+\frac{v^2}{2}+gz)\underline{v}\cdot d\underline{A}=-\oiint_A p\underline{v}\cdot d\underline{A}+P+\dot{Q}$$

$$\oiiint_V \frac{\partial}{\partial t}[\rho(e+\frac{v^2}{2}+gz)]dV+\oiint_A \rho(e+\frac{p}{\rho}+\frac{v^2}{2}+gz)\underline{v}\cdot d\underline{A}=P+\dot{Q}\text{..........(2.8)}$$

تطبيق على سريان رتيب أحادي البعد:

رتابة السريان تعني أن الحد الأول في المعادلة (2.8) يساوي صفر، و لا انتقال للكتلة عبر الأسطح (3) و (4).

وبذلك تُختزل المعادلة إلى الصورة



الشكل 2.5

$$-\rho_1(e_1+\frac{p_1}{\rho_1}+\frac{v_1^2}{2}+gz_1)v_1A_1+\rho_2(e_2+\frac{p_2}{\rho_2}+\frac{v_2^2}{2}+gz_2)v_2A_2=P+\dot{Q}$$

بالاستعانة بمعادلة حفظ الكتلة للسريان الرتيب أحادي البعد (2.5)

$$\rho_1 \boldsymbol{v}_1 A_1=\rho_2 \boldsymbol{v}_2 A_2=\dot{m}$$

$$\dot{m}(e_1+\frac{p_1}{\rho_1}+\frac{v_1^2}{2}+gz_1)+P+\dot{Q}=\dot{m}(e_2+\frac{p_2}{\rho_2}+\frac{v_2^2}{2}+gz_2)$$

$$\frac{e_1}{g}+\frac{p_1}{\rho_1 g}+\frac{v_1^2}{2g}+z_1+\frac{P}{\dot{m}g}+\frac{\dot{Q}}{\dot{m}g}=\frac{e_2}{g}+\frac{p_2}{\rho_2 g}+\frac{v_2^2}{2g}+z_2 \quad ....(2.9)$$

في كثير من التطبيقات الهندسية يمكن تجاهل انتقال الحرارة $\dot{Q}=0$

و تجاهل التغير في درجة الحرارة $T_1=T_2\,,\quad e_1=e_2$

ويمكن اعتبار السريان لا انضغاطي $\rho_1=\rho_2=\rho$

فتصبح المعادلة (2.9)

$$\frac{p_1}{\rho_1 g}+\frac{v_1^{\;2}}{2g}+z_1+\frac{P}{\dot{m}g}=\frac{p_2}{\rho_2 g}+\frac{v_2^{\;2}}{2g}+z_2 \quad \text{.................................}(2.10)$$

في حال أن القدرة P موجبة فإنها تمثل مضخة و إذا كانت سالبة فتمثل عنّفة.

في حال عدم وجود مضخة أو عنفة بين المقطعين (1) و (2) تصبح المعادلة (2.10)

$$\frac{p_1}{\rho g}+\frac{v_1^{\;2}}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^{\;2}}{2g}+z_2 \quad = \quad \text{السمت الكلي} \;\; ....(2.11)$$

أي: السمت الكلي = سمت الرفع + سمت السرعة + سمت الضغط

مثال

يُعرّف الآتي عن وحدة ضخ ترفع الماء من النيل إلى أعلى الجرف:

الرفع: 8m

معدل السريان الحجمي 15 l/s

قطر الأنبوب صعيد المضخة: 154mm

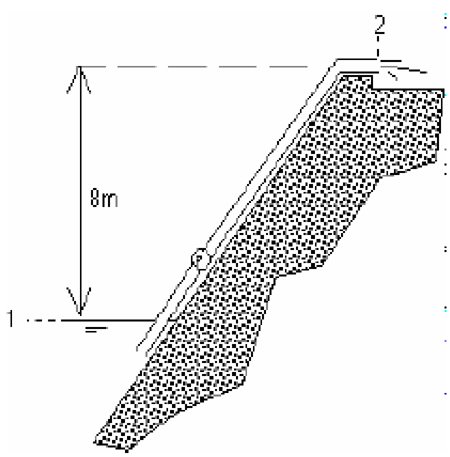
قطر الأنبوب سافل المضخة: 102mm

كثافة الماء: $1000 kg\backslash m^3$

المطلوب حساب:

(أ) السرعة صعيد وسافل المضخة

(ب) القدرة الخارجة من المضخة إذا اعتبرنا السريان لا لزجي.



الشكل (2.6)

(أ) معادلة حفظ الكتلة (2.5) للسريان اللاإنضغاطى تُعطي

$$\boldsymbol{v}_u \,.\, A_u = \boldsymbol{v}_d .\, A_d = \dot{V} = 0.015\ m^3/s$$

$$v_u = \frac{0.015}{\frac{\pi}{4}(0.154)^2} = 0.81 m/s$$

$$v_d = \frac{0.015}{\frac{\pi}{4}(0.102)^2} = 1.84 m/s$$

حيث اللاحقة u تعني صعيد المضخة و اللاحقة d تعني سافل المضخة.

(ب) معادلة الطاقة لهذه الحالة (2.10)

$$\frac{p_1}{\rho g} + \frac{v_1^{\,2}}{2g} + z_1 + \frac{P}{\dot{m}\, g} = \frac{p_2}{\rho g} + \frac{v_2^{\,2}}{2g} + z_2$$

$$P = \dot{m}\, g \left[\frac{p_2 - p_1}{\rho g} + \frac{v_2^{\,2} - v_1^{\,2}}{2g} + (z_2 - z_1) \right]$$

المقطعان (1) و (2) مفتوحان للجو و يعني ذلك

$$p_1 = p_2 = p_a$$

$$p_2 - p_1 = 0$$

كما أن $z_2 - z_1 = 8$

السطح (1) سطح النيل: سرعة نقصانه صفر !

$$v_1 = 0 \ ,\ v_2 = v_d$$

معدل سريان الكتلة $\dot{m}$

$$\dot{m} = \rho\, \dot{V} = 1000(0.015) = 15.0\ kg/s$$

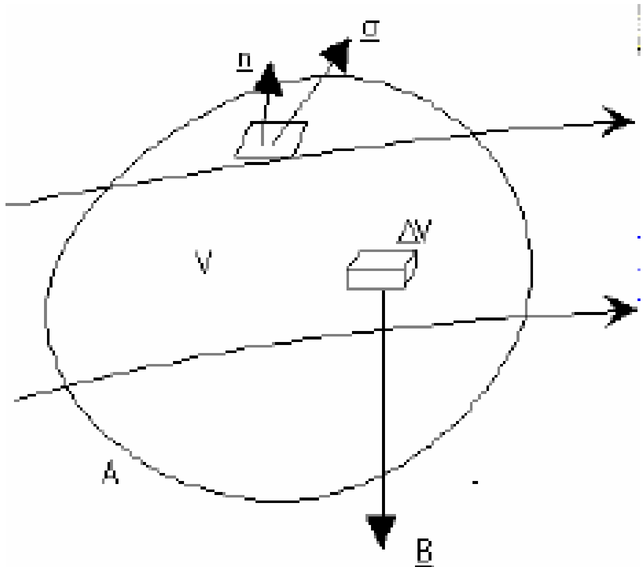
وتصبح المعادلة

$$\mathbf{P = (15.0)(9.81)\ [\ \frac{(1.84)^2}{2(9.81)} + 8\ ] = 1203W}$$

القدرة الخارجة = **1.2 kW**

2.6 حفظ كمية التحرك (momentum conservation)

الشكل 2.4



يستمد هذا القانون من قانون نيوتن الثاني (Second Newtonian Law) للحركة مطابقاً على حخم التحكمي:
"معدل تراكم كمية التحرك داخل الحجم التحكمي مضافاً اليه خالص معدل سريان كمية التحرك إلى خارج الحجم التحكمي بإنتقال الكتلة يعادل مجموع القوى المؤثرة على المائع."

$$\frac{\partial}{\partial t}\oiiint_V (\rho \underline{v})dV + \oiint_A \rho \underline{v}(\underline{v}.d\underline{A}) = \oiiint_V \underline{B}dV + \oiint_A \underline{\sigma}dA$$

$$\oiiint_V \frac{\partial}{\partial t}(\rho \underline{v})dV + \oiint_A \rho \underline{v}(\underline{v}.d\underline{A}) = \oiiint_V \underline{B}dV + \oiint_A \underline{\sigma}dA \quad ....(2.12)$$

نسترجع هنا أن الإجهاد $\underline{\sigma}$ يساوي مجموع المتجهين $p\underline{n}$ - و $\underline{\tau}$. كما أن $\underline{B}$ هي القوة الجسمية على وحدة حجمية و تتمثل في الأحوال الأعم في قوة الجاذبية على وحدة حجمية أي $\underline{B} = -\rho g \underline{k}$.

2.7 تلخيص المعادلات الاساسية (governing equations) لديناميك الموائع مع ملاحظات

2.7.1 معادلات السريان اللزجي (viscous flow) دون النظر الى تفاعلات الكيميائية (without considering chemical reactions)

Viscous flow: a flow which includes the dissipative, transport phenomena of viscosity and thermal conduction. The additional transport phenomenon of mass diffusion is not included because we are	السريان اللزجي هو الذي من ضمنه .... dissipative, transport phenomena of)

limiting our considerations to a homogenous, non-chemically reacting gas. Combustion for example is a flow with a chemical reaction. If diffusion were to be included, there would be additional continuity equations – the species continuity equations involving mass transport of chemical species *i* due to a concentration gradient in the species. Moreover the energy equation would have an additional term to account for energy transport due to the diffusion of species. With the above restrictions in mind, the governing equations for an unsteady, three-dimensional, compressible, viscous flow are:	(viscosity)، و .... (thermal conduction)
Continuity equations (Non-conservation form – [Wendt 2009], Eq.2.18)	**معادلات الاستمرارية** **(بالشكل الغير محافظي)**
$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$	

<table>
<tr><td>(Conservation form – [Wendt 2009], Eq. 2.27)</td><td></td></tr>
<tr><td colspan="2">$\frac{\partial \rho}{\partial t} + \nabla(\rho \cdot \vec{V}) = 0$</td></tr>
<tr><td>Equation [Wendt 2009], (2.18) is the continuity equation in non-conservation form. Note that:
1. By applying the model of an infinitesimal fluid element, we have obtained Eq. [Wendt 2009], (2.18) directly in partial differential form.
2. By choosing the model to be moving with the flow, we have obtained the non-conservation form of the continuity equation, namely Eq. [Wendt 2009], (2.18).
Equation [Wendt 2009], (2.27) is the continuity equation in conservation form. Note that:</td><td></td></tr>
</table>

[6] Integral form of the continuity equation: ([Wendt 2009], Eq. 2.23) $\frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \, d\mathcal{V} + \oiint_{S} \rho \vec{V} \cdot \vec{dS} = 0$

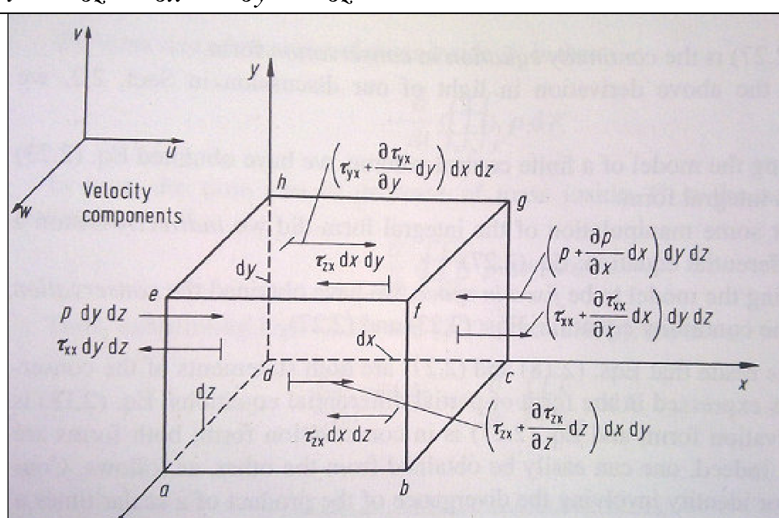
1. By applying the model of an *finite control volume*, we have obtained Eq. [Wendt 2009], (2.23) *directly* in integral form.[6] Only after some manipulation of the integral form the partial differential form, namely Eq. [Wendt 2009], (2.27), is obtained. 2. By choosing the model to be *fixed in space*, we have obtained the conservation form of the continuity equation, namely Eqs. [Wendt 2009], (2.13) and (2.27).	
Momentum equations (Non-conservation form – [Wendt 2009], Eqs. 2.36a-c)	**معادلات كمية التحرك**

x-component: $\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$

y-component: $\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$

z-component: $\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$

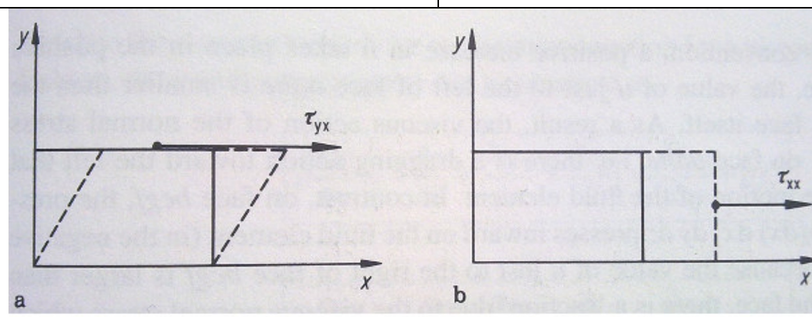
[Wendt 2009], Fig.2.5: Infinitesimally small, moving fluid element. Only the forces in the x direction are shown.



Total force in the x-direction: F_x [Wendt 2009], S.28 Def. of body forces and surface forces: *Body forces*, which act directly on the volumetric mass of the fluid element. **Examples: gravitational, electric and magnetic forces.** • Def.: body force on the fluid element acting	F_x هي القوة الاجمالية في اتجاه x هناك نوعين من القوة في هذا الايطار: 1. **<u>قوات جسمية</u>** التي تتفاعل مباشرةً على الكتلة الحجمية للعضو مائعي (fluid element). و امثلة هي: القوة الجاذبية والكهروبائية

in the x-direction $= \rho f_x (dxdydz)$.	والمغناطسية.
Surface forces, which act directly on the surface of the fluid element. They are due to only two sources: (a) pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (b) the shear and normal stress distributions acting on the surface, also imposed by the outside fluid "tugging" or "pushing" on the surface by means of friction.	2. **قوات سطحية** التي تتفاعل مباشرة على سطع العضو المائعي.

[Wendt 2009], Fig.2.6: Illustration of shear and normal stresses



(Conservation form – [Wendt 2009], Eqs. 2.42a-c)	

x-component: $$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \rho f_x$$

y-component: $$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \rho f_y$$

z-component: $$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho f_z$$

Energy equation (Non-conservation form – [Wendt 2009], Eq. 2.52)	**معادلة الطاقة**

$$\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$

$$-\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x}$$

$$+\frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y}$$

$$+\frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho \vec{f} \cdot \vec{V}$$

(Conservation form – [Wendt 2009], Eq. 2.64)	

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\vec{V}\right)\right]$$ $$=\rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$ $$+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\frac{\partial(u\tau_{xx})}{\partial x}$$ $$+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}$$ $$+\frac{\partial(v\tau_{zy})}{\partial z}+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}+\rho\vec{f}\cdot\vec{V}$$

2.7.2 معادلات السريان الا لزجي (inviscous flow) دون النظر الى تفاعلات الكيميائية (without considering chemical reactions)

Here are the viscous terms of the above equations dropped.	

2.7.3 تعليقات على المعادلات الاساسية

Surveying the above governing equations, several comments and observations can be made: 1. They are coupled system of non-linear partial differential equations, and hence are very difficult to solve analytically. To date, there is no general closed-form solution to these equations. 2. For the momentum and energy equations, the difference between the non-conservation and conservation forms of the equation is just the left-hand side. 3. Note that the conservation form of the equationscontain terms on the left-hand side which include the divergence of some quantity, such as $\nabla\cdot(\rho\cdot\vec{V})$, $\nabla\cdot(\rho u\vec{V})$, etc. For this reason, the conservation form of the governing equations is sometimes called the *divergence form*. 4. The normal and stress terms in these equations are functions of the velocity gradients, as given by [Wendt 2009], Eqs. (2.43a-f). 5. The system contains five equations in terms of six	**اذا نتأمل المعادلات الاساسية، نستطيع ان نقول التالى:** 1. هي ممجموعة مزواجة من

unknown flow-field variables, ρ, p, u, v, w, e. In aerodeynamics, it is generally reasonable to assume the gas is a perfect gas (which assumes that intermolecular forces are negligible). For a perfect gas, the equation of state is $p = \rho RT$, where R is the specific gas constant. This provides a sixth equation, but it also introduces a seventh unknown, namely temperature, T. A seventh equation to close the entire system must be a thermodynamic relation between state variables. For example, e = e(T,p) For a calorically perfect gas (constant specific heats), this relation would be $e = c_v T$ where c_v is the specific heat at constant volume. 6. Historically, the momentum equations for a viscous flow are called the ***Navier-Stokes equations***. However, in modern CFD literature, "a Navier-Stokes solution" simply means a solution of a *viscous flow problem* using *full governing equations (including continuity as well as energy and momentum).*	

2.7.4 الاحوال الجدارية (**boundary conditions**)

The boundary conditions, and sometimes the initial conditions, dictate the particular solutions to be obtained from the governing equations. (This makes the difference for example between the flow over a Boing 757 or past a wind mill, although the equations are the same). For a viscous fluid, the boundary condition on a surface assumes no relative velocity between the surface and the gas immediately at the surface. This is called the *no-slip* condition. If the surface is stationary, then $u = v = w = 0$ at the surface (for a viscous flow) For an inviscid fluid, the flow slips over the surface (there is	

no friction to promote its 'sticking' to the surface); hence, at the surface, the flow must be tangent to the surface. $\vec{V}\cdot\vec{n}=0$ at the surface (for a inviscid flow) where $\vec{n}$ is a unit vector perpendicular (that means orthogonal) to the surface. The boundary conditions elsewhere in the flow depend on the type of problem being considered, and usually pertain to inflow and outflow boundaries at a finite distance from the surfaces, or an 'infinity' boundary condition infinitely far from surface. The boundary conditions discussed above are physically boundary conditions in nature. In CFD we have a additional concern, namely the proper numerical implementation of the boundary conditions.	

2.8 اشكال للمادلات الاساسية تلائم مع CFD: ملاحظات على الشكل التحفظي (conservation form)

نستطيع ان نكتب مجموعة المعادلات الاساسية بالشكل التحفظي (conservation form) بالشكل العام التالي:

$$\boxed{\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=J}$$

[Wendt], Eq. 2.65

حيث

$U=\begin{Bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho(e+V^2/2)\end{Bmatrix}$	$H=\begin{Bmatrix}\rho w\\ \rho uw-\tau_{zx}\\ \rho vw-\tau_{zy}\\ \rho w^2+p-\tau_{zz}\\ \rho(e+V^2/2)w+pw-k\dfrac{\partial T}{\partial z}-u\tau_{zx}-v\tau_{zy}-w\tau_{zz}\end{Bmatrix}$
$F=\begin{Bmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho vu-\tau_{xy}\\ \rho wu-\tau_{xz}\\ \rho(e+V^2/2)u+pu-k\dfrac{\partial T}{\partial x}-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}\end{Bmatrix}$	$J=\begin{Bmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho(uf_x+v\rho f_y+w\rho f_z)+p\dot{q}\end{Bmatrix}$
$G=\begin{Bmatrix}\rho v\\ \rho uv-\tau_{yx}\\ \rho v^2+p-\tau_{yy}\\ \rho wv-\tau_{yz}\\ \rho(e+V^2/2)v+pv-k\dfrac{\partial T}{\partial y}-u\tau_{yx}-v\tau_{yy}-w\tau_{yz}\end{Bmatrix}$	

In [Wendt], Eq. 2.65, the column vectors F, G, and H are called the flux terms (or flux vectors), and J represents a 'source term' (which is zero if body forces are negligible). For an unsteady problem, U is called the solution vector because the elements in $U\ (\rho, \rho u, \rho v, \text{etc.})$ are the dependent variables which are usually solved numerically in steps of time. Please note that, in this formalism, it is the elements of U that are obtained computationally, i.e. numbers are obtained for the products $\rho, \rho u, \rho v, \rho w$ and $\rho(e+V^2/2)$. Of course, once numbers are known for these dependent variables (which includes ρ by itself), obtaining the primitive variables is simple:	في المعادلة ، الموجهات العودية F و G و H تسم الموجهات الجريانية.

$$\rho = \rho$$

$$u = \frac{\rho u}{\rho}$$

$$v = \frac{\rho v}{\rho}$$

$$w = \frac{\rho w}{\rho}$$

$$e = \frac{\rho(e+V^2/2)}{\rho} - \frac{u^2+v^2+w^2}{2}$$

For an *inviscid flow*, [Wendt et. al. 2009], Eq.(2.65) remains the same, except the elements of the column vectors are simplified. Examining the conservation form of the inviscid equations summerized in Sect. 2.7.2, we find that	لسريان لا لزجي المعادلة [Wendt et. al. 2009], Eq.(2.65) تبقى كما هي، الا ان الموجهات العامودية اصبحت ابسط. اذا تأملنا الشكل التحفظي للمعادلات اللا لزجية في باب 2.7.2 نجد ان

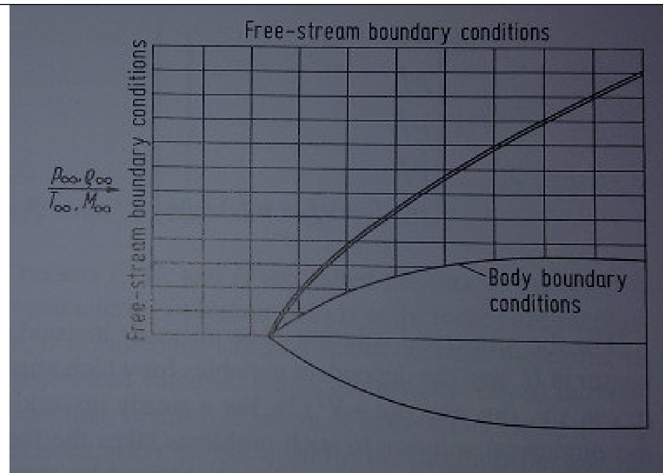
$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho(e + V^2/2) \end{Bmatrix}$$	$$F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho vu \\ \rho wu \\ \rho u(e + V^2/2)u + pu \end{Bmatrix}$$
$$G = \begin{Bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho wv \\ \rho v(e + V^2/2) + pv \end{Bmatrix}$$	$$H = \begin{Bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ \rho w(e + V^2/2) + pw \end{Bmatrix}$$ $$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(uf_x + v\rho f_y + w\rho f_z) + p\dot{q} \end{Bmatrix}$$
For the numerical solution of an unsteady inviscid flow, once again the solution vector is U, and the dependent variables for which numbers are directly obtained are products $\rho, \rho u, \rho v, \rho w$ and $\rho(e + V^2/2)$. For a steady inviscid flow, $\partial U/\partial t = 0$. Frequently, the numerical solution to such problems takes the form of 'marching' techniques; for example, if the solution is being obtained by marching in the x-direction, then [Wendt et. al. 2009], Eq.(2.65) can be written as	

	$$\frac{\partial F}{\partial x} = J - \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}$$	[Wendt], Eq. 2.66

Here, F becomes the 'solution vector', and the dependent variables for which numbers are obtained are $\rho, \rho u, \rho v, \rho w$ and $\rho(e + V^2/2)$. From these dependent variables, it is still possible to obtain the primitive variables, although the algebra is more complex than in the previously discussed case. Notice that the governing equations when written in the form of [Wendt et. al. 2009],	هنا F يصبح الموجه المحلول.

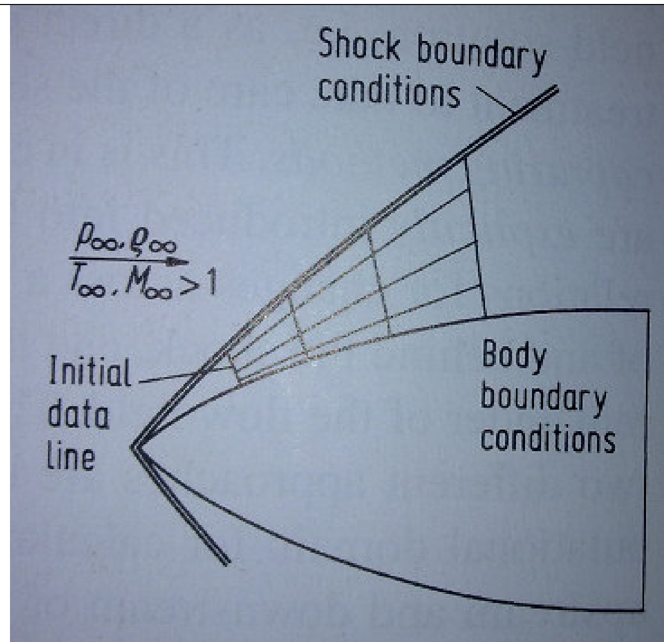
Eq.(2.65), have no flow variables outside the single x,y,z, and t derivates. Indeed, the terms in [Wendt et. al. 2009], Eq.(2.65) have everything buried inside these derivates. The flow equations in the form of [Wendt et. al. 2009], Eq.(2.65) are said to be in strong conservation form. In contrast, examine the forms [Wendt et. al. 2009], Eq.(2.42a,b and c) and [Wendt et. al. 2009], Eq.(2.64). These equations have a number of x,y and z derivates exipliticly appearing on the right –hand side. These are the *weak conservation* form of the equations.	
The form of the governing equations giving by Eq. (2.65) is popular in CFD; let us explain why. In flow fields involving shock waves, there are sharp, discontinuous changes in the primitive flow-field variables p, p, u, T, etc., across the shocks. Many computations of flows with shocks are designed to have the shock waves appear naturally within the computational space as a direct result of the overall flow field solution, i.e. as a direct result of the general algorithm, without any special treatment to take care of the shocks themselves. Such approaches are called shock capturing methods. This is in contrast to the alternate approach, where shock waves are explicitly introduced into the flow-field solution, the exact Rankine-Hugoniot relations for changes across a shock are used to relate the flow immediately ahead of and behind the shock, and the governing flow equations are used to calculate the remainder of the flow field. This approach is called the shock-fitting method. These two different approaches are illustrated in Figs. 2.8 and 2.9. In Fig.2.8, the computational domain for calculating the supersonic flow over the body extends both upstream and downstream of the nose. The shock wave is allowed to form within the computational domain as a consequence of the general flow-field algorithm,	

[Wendt et.al.2009], Fig.2.8: Mesh for the shock-capturing approach



without any special shock relations being introduced. In this manner, the shock wave is ‘captured’ within the domain by means of the computational solution of the governing partial differential equations. Therefore, Fig. 2.8 is an example of the shock-capturing method. In contrast, Fig. 2.9 illustrates the same flow problem, except that now the computational domain is the flow between the between the shock and the body. The shock wave is introduced directly into the solution as an explicit discontinuity, and the standard oblique shock relations (the Rankine-Hugoniot relations) are used the freestream supersonic flow ahead of the shock to the flow computed by the partial differential equations downstream of the shock. Therefore, Fig. 2.9 is an example of the shock-fitting method. There are advantages and disadvantages of both methods. For example, the shock-capturing method is ideal for complex flow problems involving shock waves for which we do not know either the location or number of shocks. Here, the shocks simply form within the computational domain as nature would have it. Moreover, this takes place without requiring any special treatment of the shock within the algorithm, and hence simplifies the computer programming. However, a disadvantage of this approach is that the shocks are generally smeared over a number of grid points in the computational mesh, and hence the numerically obtained shock thickness bears no relation what-so-ever to the actual physical shock thickness, and the precise location of the shock discontinuity is uncertain within a few mesh sizes. In contrast, the advantage of the shock-fitting method is

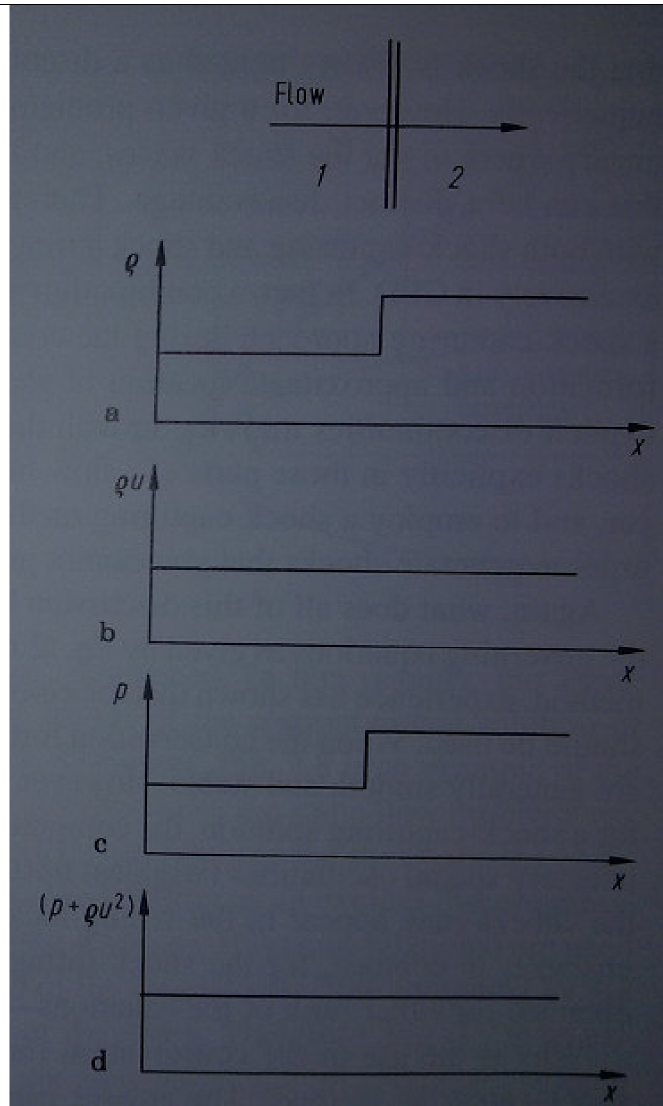
[Wendt et.al.2009], Fig.2.9: Mesh for the shock-fitting approach



That the shock is always treated as a discontinuity, and its location is well-defined numerically. However, for a given problem you have to know in advance approximately where to put the shock waves, and how many there are. For complex flows, this can be a distinct disadvantage. Therefore, there are pros and cons associated with both shock-capturing and shock-fitting methods, and both have been employed extensively in CFD. In fact, a combination of these two methods is used to predict the formation and approximate location of shocks, and then these shocks are fit with explicitly in those parts of a flow field where you know in advance they occur, and to employ a shock-capturing method for the remainder of the flow field in order to generate shocks that you cannot predict in advance.

Again, what does all of this discussion have to do with the conservation form of the governing equations as given by Eq. (2.65)? Simply this. For the shock-capturing method, experience has shown that the conservation form of the governing equations should be used. When the conservation form is used, the computed flow-field results are generally smooth and stable. However, when the non-conservation form is used for a shock-capturing solution, the computed flow-field results usually exhibit unsatisfactory spatial oscillations (wiggles) upstream and downstream of the shock wave, the shocks may appear in the wrong location and the solution may even become unstable. In contrast, for the shock-fitting method, satisfactory results are usually obtained for either form of the

equations-conservation or non-conservation.	
Why is the use of the conservation form of the equations so important for the shock-capturing method? The answer can be see by considering the flow across a normal shock wave, as illustrated in Fig. 2.10. Consider the density distribution across the shock, as sketched in Fig. 2.10(a). Clearly, there is a discontinuous increase in p across the shock. If the non-conservation from of the governing equations were used to calculate this flow, where the primary dependent variables are the primitive variables such as p and p, then the equations would see a large discontinuity in the dependent variable p. This in turn would compound the numerical errors associated with the calculation of p. On the other hand, recall the continuity equation for a normal shock wave (see Refs.[1,3]):	
$\rho_1 u_1 = \rho_2 u_2$ (2.67)	
From Eq. (2.67), the *mass flux*, ρu, is constant across the shock wave, as illustrated in Fig. 2.10(b). The conservation form of the governing equations uses the product ρu as a dependent variable, and hence the conservation form of the equations see no discontinuity in this dependent variable across the shock wave. In turn, the numerical accuracy and stability of the solution should be greatly enhanced. To reinforce this discussion, consider the momentum equation across a normal shock wave [1,3]:	
$\rho_1 + \rho_1 u_1^2 = \rho_2 + \rho_2 u_2^2$ (2.68)	
As show in Fig. 2.10(c), the pressure itself is discontinuous across the shock ; however, from Eq. (2.68) the flux variable ($\rho + \rho u^2$) is constant across the shock.	

[Wendt et. al. 2009], Fig.2.10: Variation of flow properties through a normal shock wave	

This is illustrated in Fig. 2.10(d). Examining the inviscid flow equations in the conservation form given by Eq. (2.65), we clearly see that the quantity ($\rho + \rho u^2$) is one of the dependent variables. Therefore, the conservation form of the equations would see no discontinuity in this dependent variables across the shock. Although this example of the flow across a normal shock wave is somewhat simplistic, it serves to explain why the use of the conservation form of the governing equations are so important for calculations using the shock-capturing method. Because the conservation form uses flux variables as the dependent variables, and because the changes in these flux variables are either zero or small across a shock wave, the numerical quality of a shock-capturing method will be enhances by

the use of the conservation form in contrast to the non-conservation form, which uses the primitive variables as dependent variables. In summary, the previous discussion is one of the primary reasons why CFD makes a distinction between the two forms of the governing equations-conservation and non-conservation. And this is why we have gone to great lengths in this chapter to derive these different forms, and why we should be aware of the differences between the two forms.	

2.9 مراجع **References**

Anderson, John D., Jr.., *Fundamentals of Aerodynamics*, 2nd Edition McGraw-Hill, New York, 1991.

Liepmann, H.W. and Roshko, A., *Elements of Gasdynamics*, Wiley, New York, 1957.

Anderson, J.D., Jr., *Modern Compressible Flow: With Historical Perspective*, 2nd Edition McGraw-Hill, New York, 1990.

Bird, R.B,. Stewart, W.E. and Lightfoot, E.N. *Transport Phenomena*, 2nd edition, Wiley, 2004.

Kutler, P., 'Computation of Three-Dimensional, Inviscid Supersonic Flows,´ in H.J. Wirz (ed.), *Progress in Numerical Fluid Dynamics*, Springer-Verlag, Berlin, 1975, pp. 293-374.

3 سراين لا انضغاطية و لا لزجية (Incompressible Inviscid Flows) : طرق حسابية معتمدة على مؤطرات النبع و الدوامة (Source and Vortex Panel Methods)

3.1 مدخل

في هذا الفصال سننظر ان شاء الله الى التحليل العددي (numerical analysis) لسراين (flows) لا انضغاطية (incompressible) و لا لزجية (inviscid). مأبدئياً يمكن ان يستخدم طريقة الفرق المحدود (finite-difference method) – التي ستناقش في ما بعد ان شاء الله– لحل هذا النوع من السراين. ولكن يوجد طرق اخرى تأدي عدة الى حلول اكثر مناسبة لسراين لا انضغاطية (incompressible) و لا لزجية (inviscid).

هذا الفصل يناقش احد هذه الطرق – المساة طرق حسابية معتمدة على مؤطرات النبع و الدوامة (Source and Vortex Panel Methods). هذه الطرق اصبحت هي الطرق المقياسية والمعتمد عليها عادة في الشركات التي تصنع الطياران و هذا منذ العقد 1960

طرق المؤطرات هي طرق حسابية عددية (numerical methods) تحتاج الى قوة حسابية ضخمة و لذلك كومبيوترات سريعة.

3.2 بعض الاوجهة الاساسية لسريان لا انضغاطي و لا لزجي

السريان الغير انضغاطي (incompressible flow) هو سريان بكثافة (density) ثابتة ($\rho = const.$).

تصور عضو مائع (fluid element) بكتلة ثابتة (m = const.) يجري في سريان غير انضغاطي (incompressible flow) في موازاة خط انسياب (streamline). لأن الكثافة ثابتة فبالتالي الحجم (volume) لهذا العضو مائعي هو ايضا ثابت (V = const.). و لأن $\nabla\vec{V}$ ($\vec{V}$ هي السرعة) يشكل التغيير لحجمي لعضو مائعي على مدار الزمان نستطيع ان نكتب:

$$\nabla\vec{V} = 0 \qquad (3.1)$$

∇ هنا الNABLA-Operator و هو علامة ملخصة لgrad و هو الgradient.

و إلى هذا فاذا العضو مائعي (fluid element) ايضاً لا يدور لما يتحرك في موازاة الخط الانسياب (streamline) فبالتالي هذا السريان (flow) يسم لا دوراني (irrotational). لهاذا النوع من السرايين، يمكن ان يعبر عن السرعة (velocity) كبوتينزيال (potential) – يُعلم بِ ϕ [7].

$$\vec{V} = \nabla\phi \qquad (3.2)$$

$$\mathrm{grad}\phi = \nabla\phi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix}$$

إذا جمعنا الآن معادلة (3.1) و (3.2) نصل الى:

$$\nabla \cdot \nabla\phi = 0$$

او،

---

[7] لمزيد من الشرح انظر ملحق أ و (Anderson 1991).

$$\boxed{\nabla^2 \phi = 0} \qquad (3.3)$$

(3.3) تسمى معادلة Laplace (Laplace's equation)، احد المعادلات المشهورة والمدروسة جيداً في مجال الفيزيك الرياضية (mathematical physics).

من معادلة (3.3) نرى ان سرايين (flows) لا انضغاطية (incompressible) و لا لزجية (inviscid) تُحَكَّم بمعادلة Laplace (Laplace's equation).

و معادلة Laplace (Laplace's equation) هي خطية (linear).

و لذلك كل عدد من حلول خصوصية لمعادلة (3.3) يمكن ان تزاد (added) مع بعض ليستنتج حل آخر.

و هذا يُري فلسفة اساسية لحل من سريان غير انضغاطي (incompressible flow) و هو ان:

تركيب معقد لسريان غير انضغاطي و لا دوراني (incompressible, irrotational flow) يمكن ان يجمع (synthesized) من سرايين اساسية (elementary flows)

بالتالي سننظر إن شاء الله الى بعض السرايين اساسية (elementary flows) التي تلائم (satisfy) مع معادلة Laplace (Laplace's equation).

Uniform flow	
$\phi = V_\infty x$	

Source flow	
$\phi = \dfrac{\Lambda}{2\pi} \ln r$	

Vortex flow	
$\phi = -\dfrac{\Gamma}{2\pi} \theta$	

In [Wendt et. al. 2009] there are two methods described which use these elementary flows:

- Non-lifting Flows Over Arbitrary Two-Dimensional Bodies: The Source Panel Method
- Lifting Flows Over Arbitrary Two-Dimensional Bodies: The Vortex Panel Method

Also the application “The Aerodynamics of Drooped Leading-Edge Wings Below and Above Stall“ is described.

4 الخصوصيات الرياضية (Mathematical Properties) لمعادلات ديناميك الموائع (Fluid Dynamic Equations)

4.1 مدخل (Introduction)

المعادلات الاساسية من ديناميك الموائع التي استخلصت في الباب الثاني (Chapter 2) هي اما في الشكل التفاضلي (differential form) او الشكل التكاملي (integral form).

امثلة:

Integral form of the continuity equation.

Eq. 2.23

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \iint_S \rho \vec{V} \cdot \vec{dS} = 0$$

Partial differential form of the momentum equations

Momentum equations (Non-conservation form – [Wendt 2009], Eqs. 2.36a-c)	**معادلات كمية التحرك**
x-component: $\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$ y-component: $\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$ z-component: $\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$	

The governing equations in the form of partial differential forms (as [Wendt et.al.2009], Eqs. 2.36 a-c, see Chapter 2.7) are by far the most prevalent form used in computational fluid dynamics (CFD). Therefore, before studying numerical methods for the solution of these equations, it is useful to examine some mathematical properties of partial differential equations themselves. Any valid numerical solution of the equations should exhibit the property of obeying the general mathematical properties of the governing equations. Examine the governing equations of fluid dynamics as derived in Chap.2. Note that in all cases the *highest order* derivates occur linearly, i.e. there are no products or	

exponentials of the highest order derivates – they appear by themselves, multiplied by coefficients which are functions of the dependent variables themselves. Such a system of equations is called a *quasilinear system*. For example, for inviscid flows, examining the equations in Sect. 2.7.2 we find the highest order derivates are first order and all of them appear linearly. For viscid flows, examining the equations in Sect. 2.7.1 we find the highest order derivates are second order and all of them appear linearly. For this reason, in the next section, let us examine some properties of a system of quasilinear partial differential equations. In the process we will establish a classification of three types of partial differential equations – all three of which are encountered in fluid dynamics.	

4.2 تصنيف المعادلات التفاضلية الجزئية (Classification of Partial Differential Equations)

For simplicity, let us consider a fairly simple system of quasilinear equations. They will not be the flow equations, but they are similar in some respects. Therefore, this section serves as a simplified example. Consider the system of quasilinear equations given below:	
$a_1\frac{\partial u}{\partial x}+b_1\frac{\partial u}{\partial y}+c_1\frac{\partial v}{\partial x}+d_1\frac{\partial v}{\partial y}=f_1$ $a_2\frac{\partial u}{\partial x}+b_2\frac{\partial u}{\partial y}+c_2\frac{\partial v}{\partial x}+d_2\frac{\partial v}{\partial y}=f_2$	[Wendt et. al. 2009], Eq. (4.1a) [Wendt et. al. 2009], Eq. (4.1b)
where u and v are the dependent variables, functions of x and y, and the coefficients $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, f_1$ and f_2 can be functions of x, y, u and v. Consider any point in the xy-plane. Let us seek the lines (or directions) through this point (if any exist) along which the *derivates* of u and v are indeterminant, and across which may be discontinuous. Such lines are called characteristic lines. To find such lines, we assume that are continuous, and hence	

since $u = u(x,y)$: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$	[Wendt et. al. 2009], Eq. (4.2a)
since $v = v(x,y)$: $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$	[Wendt et. al. 2009], Eq. (4.2b)
Equations [Wendt et. al. 2009], Eq. (4.1a and b) and [Wendt et. al. 2009], Eq. (4.2a and b) constitute a system of four linear equations with four unknowns ($\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$, and $\partial v/\partial y$). These equations can be written in matrix form as	
$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix}$	[Wendt et. al. 2009], Eq. (4.3)
Let $[A]$ denote the coefficient matrix.	
$[A] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix}$	
Moreover, let $\|A\|$ be the determinant of $[A]$. From Cramer's rule, if $\|A\| \neq 0$, then unique solutions can be obtained for $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$, and $\partial v/\partial y$. On the other hand, if $\|A\| = 0$, then $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x$, and $\partial v/\partial y$ are, at best, indeterminant. We are seeking the particular directions in the xy-plane along which these derivates of u and v and indeterminant. Therefore, let us set $\|A\| = 0$, and see what happens.	
$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0$	
Hence	لذلك
$(a_1c_2 - a_2c_1)(\mathrm{dy})^2 - (a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)(\mathrm{d}x)(\mathrm{dy}) + (b_1d_2 - b_2d_1)(\mathrm{d}x)^2 = 0$ [Wendt et. al. 2009], Eq. (4.4)	
Divide [Wendt et. al. 2009], Eq. (4.4) by $(dx)^2$.	

$$(a_1c_2 - a_2c_1)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - (a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)\frac{\mathrm{d}y}{\mathrm{d}x} + (b_1d_2 - b_2d_1) = 0$$

[Wendt et. al. 2009], Eq. (4.5)

Equation (4.5) is a quadratic equation in dy/dx. For any point in the xy-plane, the solution of Eq. (4.5) will give the slopes of the lines along which the derivatives of u and v are indeterminant. These lines in the xy space along are called characteristic lines fo the system of equations given by Wendt et. al. 2009], Eq. (4.1a and 4.1b).

In Eq. (4.5), let

$$a = (a_1c_2 - a_2c_1)$$
$$b = -(a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)$$
$$c = (b_1d_2 - b_2d_1)$$

Then Eq. (4.5) can be written as

$$a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right)^2 + c = 0$$ [Wendt et. al. 2009], Eq. (4.6)

Hence from the quadratic formula:

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ [Wendt et. al. 2009], Eq. (4.7)

Equation (4.7) gives the direction of the characteristic lines through a given xy point. These lines have a different nature, depending on the value of the discriminant in Eq. (4.7). Denote the dicriminant by D.

$$D = b^2 - 4ac$$ [Wendt et. al. 2009], Eq. (4.8)

The characteristic lines may be real and distinct, real and equal, or imaginary, depending on the value of D. Specially:

If D>0: Two real and distinct lines exist through each point in the xy-plane. When this is the case, the system of equations given by [Wendt et. al. 2009], Eqs. (4.1 a and b) is called *hyperbolic*.

If D=0: One real characteristic exists. Here the system of equations given by [Wendt et. al. 2009], Eqs. (4.1 a and b) is called *parabolic*.

If D<0: The characteristic lines are imaginary. Here the system of equations given by [Wendt et. al. 2009], Eqs. (4.1 a and b) is called *elliptic*.	
The classification of quasilinear PDEs as either *elliptic, parabolic* or *hyperbolic* is common in the analysis of such equations. These three classes of equations have totally different behaviour. The origin of the words *elliptic, parabolic* and *hyperbolic* is simply a direct analogy with the case for conic sections. The general equations for a conic section from analytic geometry is	
$ax^2 + bxy + cy^2 + dx + ey + f = 0$	
Where, if $b^2 - 4ac > 0$, the conic is a hyperbola $b^2 - 4ac = 0$, the conic is a parabola $b^2 - 4ac < 0$, the conic is a ellipse	
We note, that for hyperbolic PDEs, the fact, that two real and distinct characteristics exist, allows the development of a method for the ready solution of these equations. If we return to [Wendt et. al. 2009], Eq. (4.3), and actually attempt to solve for, say $\partial u / \partial y$, using Cramer's rule, we have	
$\partial u / \partial y = \frac{\|N\|}{\|A\|} = \frac{0}{0}$	
where the numerator determinant is	
$\|N\| = \begin{vmatrix} a_1 & f_1 & c_1 & d_1 \\ a_2 & f_2 & c_2 & d_2 \\ dx & du & 0 & 0 \\ 0 & dv & dx & dy \end{vmatrix}$ [Wendt et. al. 2009], Eq. (4.9)	
The reason why $\|N\|$ must be zero is that $\partial u / \partial y$ is indeterminant, of the form 0/0. Since $\|A\|$ has already been made to zero, then $\|N\|$ must be zero to allow $\partial u / \partial y$ to be indeterminant. The expansion of [Wendt et. al. 2009], Eq. (4.9) will lead to equations involving the flow field variables which are *ordinary differential equations*, and in some cases are algebraic equations; these equations obtained from [Wendt et. al. 2009], Eq. (4.9) are called the *compatibility* equations. They hold only along	

the characteristic lines. This is the essence of solving the original hyperbolic PDE: simply integrate simpler, ordinary differential equations (the compatibility equations) along the the characteristic lines in the xy-plane. This is called the *method of characteristics*. This method is highly developed for the solution of inviscid supersonic flows, for which the system of governing flow equations is hyperbolic. The method of characteristics is a classical technique for the solution of inviscid supersonic flows, and therefor it will not be considered in this book about CFD in any detail.

4.3 General Behaviour of the different Classes of PDEs and their Relation to Fluid Dynamics

In this section we simply discuss, without proof, some of the behaviour of hyperbolic, parabolic and elliptic PDEs, and relate this behaviour to the solution of problems in fluid dynamics.

4.3.1 Hyperbolic Equations

For hyperbolic equations, information at a given point P influences only those regions between the advancing characteristics. For example, examine Fig.4.1, which is sketched for a two-dimensional problem with two independent space variables.
Point P is located at a given (x,y). Consider the left- and right-running characteristics as shown in Fig. 4.1.

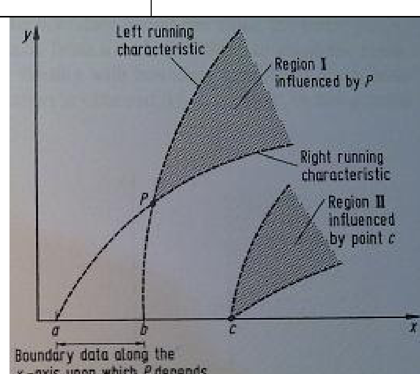
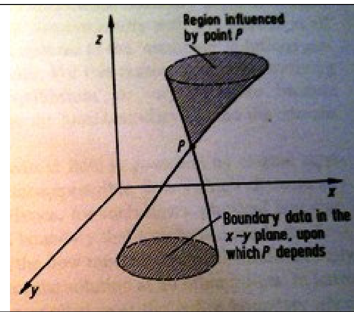


Fig. 4.1 Domain and boundaries for the solution of hyperbolic equations. Two-dimensional steady flow.

Information at point P influences only the shaded region – the region labelled I between the two advancing characteristics through P.

This has a collorary effect on boundary conditions for hyperbolic equations. Assume that the x-axis is a given boundary condition for the problem, i.e. the dependent variables u and v are known along the x-axis. Then the solution can be obtained by 'marching forward' in the distance y, starting from the given boundary. However, the solution for *u* and *v* at point P will depend only on the part of the boundary between *a* and *b*, as shown in Fig.4.1. Information at point *c*, which is outside the interval *ab*, is propagated along characteristics through *c*, and influences only region II. Point P is outside region II, and hence daes not feel information from point *c*. For this reason, point P depends on only that part of the boundary which is intercepted by and included between the two retreating characteristic lines through point P, i.e. interval *ab*. In fluid dynamics, the following types of flows are governed by hyperbolic PDEs, and hence exhibit the behaviour described above: *Steady, inviscid supersonic flow*. If the flow in two-dimensional, the behaviour is like this discussed in Fig. 4.1. If the flow in three-dimensional, there are characteristic surfaces in xyz space, as sketched in Fig. 4.2. Consider point P at a given (x,y,z) location. Information at P influences the shaded volume within the advancing characteristic surface. In addition, if the x-y plane is a boundary surface, then only that portion of the boundary shown as the cross-hatched area in the x-y plane, intercepted by the retreating characteristic surface, has any effect on P. In Fig. 4.2, the dependent variables are solved by starting with the data given in the xy-plane, and 'marching' in the z-direction. For an inviscid supersonic flow problem, the general flow direction would also be the z-direction.	

Fig. 4.2 Domain and boundaries for the solution of hyperbolic equations. Three-dimensional steady flow.



Unsteady inviscid compressible flow. For unsteady one- and two-dimensional inviscid flows, the govering equations are hyperbolic, no matter whether the flow is locally subsonic or supersonic. Here, time is the marching direction. For one-dimensional unsteady flow, consider a point P an the (x,t) plane shown in Fig. 4.3. Once again, the region influenced by P is the shaded area between the two advancing characteristics through P, and the interval *ab* is the only portion of the boundary along the x-axis upon which the solution at P depends.

For two-dimensional unsteady flow, consider a point P in the (x,y,t) space as shown in Fig. 4.4. Starting with known initial data in the xy-plane, the solution 'marches' forward in time.

Fig. 4.3 Domain and boundaries for the solution of hyperbolic equations. One-dimensional steady flow.

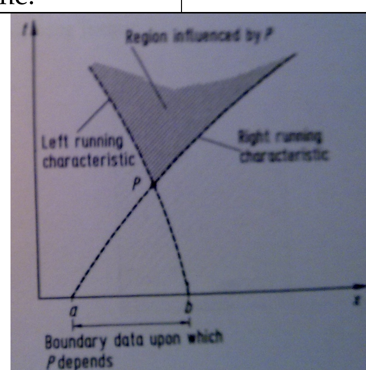
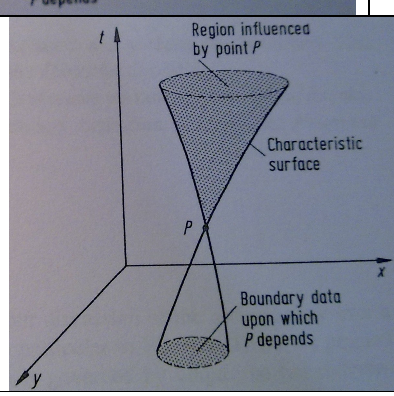
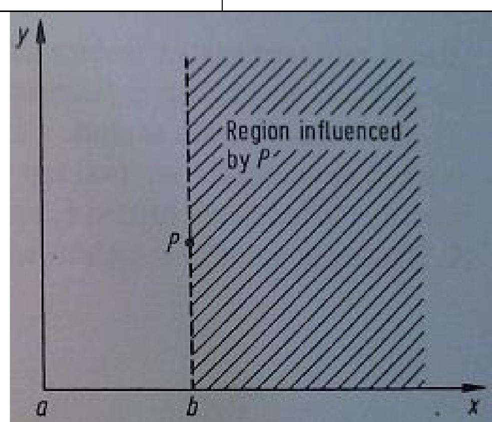


Fig. 4.4 Domain and boundaries for the solution of hyperbolic equations. Two-dimensional unsteady flow.



For parabolic equations, information at point P in the xy-plane influences the entire region of the plane to one side of P. This is sketched in Fig. 4.5, where the single characteristic line through point P is drawn. Assume the x- and y-axes are boundaries; the solution at P depends on the boundary conditions along the entire y axis, as well as on that portion of the x-axis from *a* to *b*. Solutions to parabolic equations are also 'marching' solutions; starting with boundary conditions along both the x- and y-axes, the flow-field solution is obtained by 'marching' in the general x-direction.

Fig. 4.5 Domain and boundaries for the solution of parabolic equations in two dimensions.



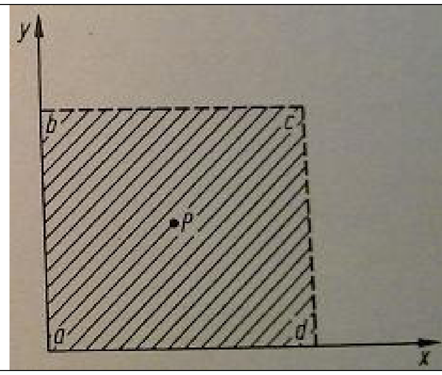
In fluid dynamics , there are reduced forms of the Navier-Stokes equations which exhibit parabolic-type behaviour. If the viscous stress terms involving derivatives with respect to x are ignored in these equations, we obtain the 'parabolized' Navier-Stokes equations, which allows a solution to march downstream in the x-direction, starting with some prescribed data along the x- and y-axes. A further reduction of the Navier-Stokes equations for the case of high Reynolds numbers leads to the well-known boundary layer equations. These boundary layer equations exhibit the parabolic behaviour shown in Fig. 4.5.

For elliptic equations, information at point P in the xy-plane influences all other regions of the domain. This is sketched in Fig. 4.6, which shows a rectangular domain. Here, the domain is fully closed, surrounded by the closed boundary *abcd*. For elliptic equations, because point P influences all points in the domain, then in turn the solution at point P is influenced by the entire closed boundary *abcd*. Therefore, the solution at point P must be carried out simultaneously with the solution at all other points in the domain. This is in be in stark contrast to the 'marching' solutions germaine to hyperbolic and parabolic equations.

In fluid dynamics steady, subsonic, inviscid flow is governed by elliptic equations. As a sub-case, this also includes incompressible flow (which theoretically implies that the Mach number is zero). Hence, for such flows, physically boundary conditions must be applied over a closed boundary that totally surrounds the flow, and the flow-field solution at all points in the flow must be obtained simultaneously, because the solution at one point influences the solution at all other points. In terms of Fig. 4.6, boundary conditions must be applied over the entire boundary *abcd*. These boundary conditions can take the following forms:

A specification of the *dependent variables* u and v along the boundary. This type of boundary conditions is called the *Dirichlet* condition.

A specification of *derivatives* of the dependent variables u and v, such as $\partial u / \partial y$ along the boundary. This type of boundary conditions is called the *Neumann* condition.

Fig. 4.6 Domain and boundaries for the solution of elliptic equations in two dimensions.		

4.3.4 بعض الملاحظات

At this stage it would be worthwhile for the student to examine the actual, closed-form solution to some linear PDE of the elliptic, parabolic and hyperbolic types. Numerous classical solutions can be found for example in Refs. [2] and [3].	

4.3.5 Well-Posed Problems

In the solution of PDEs it is sometimes easy to attempt a solution using incorrect or insufficient boundary and initial conditions. Such an 'ill-posed' problem will usually lead to spurious (مزوّر) results. Therefor we define a well-posed problem as follows: If the solution to a PDE exists and is unique, and if the solution depends continuously upon the initial and boundary conditions, then the problem is *well-posed*.	

4.3.6 References

[1] Anderson J.D., Modern Compressible Flow: With Historical Perspective, 2nd ed., 1990
[2] Hildebrand, Advanced Calculus for Applications, 1976
[3] Anderson, Tannehill and Pletcher, Computational Fluid Mechanics and Heat Transfer, 1984
[4] Moretti and Abbett, "A Time-dependent Computational Method for Blunt Body Flows", AIAA Journal, Vol.4, No.12, Dec 1966, 2136-2141

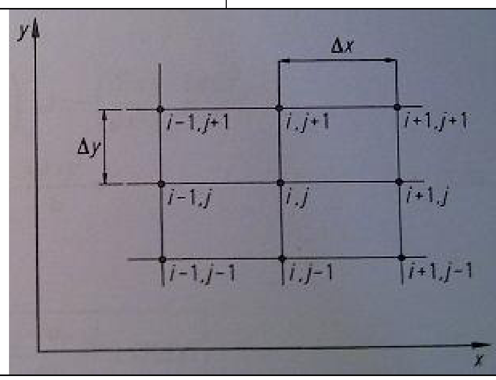
Chapter 5: Discretization of Partial Differential Equations 5

5.1 مدخل

Analytical solutions of partial differential equations involve closed-form expressions which give the variation of the dependent variables continuously throughout the domain. In contrast, numerical solutions can give answers at only *discrete points* in the domain, called *grid points*. For example, consider Fig. 5.1, which shows a section of a discrete grid in the xy-plane. For convenience, let us assume that the spacing of the grid points in the x-direction is uniform, given by Δx, and that the spacing in y-direction is also uniform, given by Δy, as shown in Fig. 5.1.In general, Δx and Δy are different. However, the vast majority of CFD applications involve numerical solutions on a grid which involves uniform spacing in each direction, because this greatly simplifies the programming of the solution, saves storage space and usually results in greater accuracy. This uniform spacing does not have to occur in physical xy space; as is frequently done in CFD, the numerical calculations are carried out in a transformed computational space which has uniform spacing in the transformed independent variables, but which corresponds to non-uniform spacing in the physical plane. These matters are discussed in Chapter 6. In any event, in this chapter we will assume uniform spacing in each coordinate direction, but not necessarily equal spacing for both directions, i.e. we will assume Δx and Δy to be constants, but that Δx does not have to equal Δy. Returning to Fig. 5.1, the grid points are identified by an index i which runs in the x-direction, and an index j which runs in the y-direction. Hence, if (i,j) is the index for point P in Fig.5.1, then the point immediately to the right of P is labelled as (i+1,j), the point direct above is (i,j+1) etc. The *method of finite differences* is widely used in CFD, and therefore most of this chapter will	

be devoted to matters concerning finite differences. The philosophy of finite differences is to replace the partial derivatives appearing in the governing equations of fluid dynamics. With algebraic difference quotients, yielding a system of algebraic equations which can be solved for the flow-field variables at the specific, discrete grid points in the flow (as shown in Fig. 5.1). Let us now proceed to derive some of the more common algebraic difference quotients used to discretize the PDEs.

Fig. 5.1 Discrete grid points



5.2 Derivation of Elementary Finite Difference Quotients

Finite difference representations of derivatives are based on Taylor's series expansions.For example, if ui, j denotes the x-component of velocity at point (i, j), then the velocity $u_{i+1,\ j}$ at point (i + 1, j) can be expressed in terms of a Taylor's series expanded about point (i, j), as follows:

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} + \cdots \qquad (5.1)$$

Equation (5.1) is mathematically an exact expression for ui+1,j if:

(a) the number of terms is infinite and the series converges,

(b) and/or Δx→0.

For numerical computations, it is impractical to carry an infinite number of terms in Eq. (5.1). Therefore, Eq. (5.1) is truncated. For example, if terms of magnitude$(\Delta x)^3$ and higher order are neglected, Eq. (5.1) reduces to

$$u_{i+1,j} \approx u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} \qquad (5.2)$$

We say that Eq. (5.2) is of second-order accuracy, because terms of order $(\Delta x)^3$ and higher have been neglected. If terms of order $(\Delta x)^2$ and higher are neglected,we obtain from Eq. (5.1),

$$u_{i+1,j} \approx u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x \qquad (5.3)$$

where Eq. (5.3) is of first-order accuracy. In Eqs. (5.2) and (5.3), the neglected higher-order terms represent the truncation error in the finite series representation. For example, the truncation error for Eq. (5.2) is

$$\sum_{n=3}^{\infty} \left(\frac{\partial^n u}{\partial x^n}\right)_{i,j} \frac{(\Delta x)^n}{n!}$$

and the truncation error for Eq. (5.3) is

$$\sum_{n=2}^{\infty} \left(\frac{\partial^n u}{\partial x^n}\right)_{i,j} \frac{(\Delta x)^n}{n!}$$

The truncation error can be reduced by:

(a) Carrying more terms in the Taylor's series, Eq. (5.1). This leads to higher-order accuracy in the representation of $u_{i+1,j}$.

(b) Reducing the magnitude of Δx.

Let us return to Eq. (5.1), and solve for $(\partial u/\partial x)_{i,j}$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \underbrace{- \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^2}{6} - \cdots}_{\text{Truncation error}}$$

or,

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \qquad (5.4)$$

In Eq. (5.4), the symbol O(Δx) is a formal mathematical notation which represents'terms of-order-of Δx'. Eq. (5.4) is more precise notation than Eq. (5.3), which involves the 'approximately equal' notation; in Eq. (5.4) the order of magnitude of the truncation error is shown explicitly by the O notation. We now identify the firstorder-accurate difference representation for the derivative $(\partial u/\partial x)_{i,j}$ expressed by Eq. (5.4) as a first-order forward difference, repeated below

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \qquad \text{(5.4 repeated)}$$

Let us now write a Taylor's series expansion for $u_{i-1,j}$, expanded about $u_{i,j}$.

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j}(-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(-\Delta x)^3}{6} + \cdots$$

or,

$$u_{i-1,j} = u_{i,j} - \left(\frac{\partial u}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{6} + \cdots \qquad (5.5)$$

Solving for $(\partial u/\partial x)_{i,j}$, we obtain

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \qquad (5.6)$$

Equation (5.6) is a first order rearward difference expression for the derivative$(\partial u/\partial x)$ at grid point (i, j).

Let us now subtract Eq. (5.5) from (5.1).

$$u_{i+1,j} - u_{i-1,j} = 2\left(\frac{\partial u}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{3} + \cdots \qquad (5.7)$$

Solving Eq. (5.7) for $(\partial u/\partial x)_{i,j}$, we obtain

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2 \tag{5.8}$$

Equation (5.8) is a second order central difference for the derivative (∂u/∂x) at grid point (i, j).To obtain a finite-difference expression for the second partial derivative $(\partial^2 u/\partial x^2)_{i,j}$, first recall that the order-of magnitude term in Eq. (5.8) comes from Eq. (5.7),and that Eq. (5.8) can be written

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} + \cdots \tag{5.9}$$

Substituting Eq. (5.9) into (5.1), we obtain

$$\begin{aligned} u_{i+1,j} = u_{i,j} &+ \left[\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} + \cdots\right]\Delta x \\ &+ \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} \\ &+ \left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} \frac{(\Delta x)^4}{24} + \cdots \end{aligned} \tag{5.10}$$

Solving Eq. (5.10) for $(\partial^2 u/\partial x^2)_{i,j}$, we obtain

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2 \tag{5.11}$$

Equation (5.11) is a second-order central second difference for the derivative $(\partial^2 u/\partial x^2)$ at grid point (i, j). Difference expressions for the y-derivatives are obtained in exactly the same fashion. The results are directly analogous to the previous equations for the x-derivatives. They are:

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y} + O(\Delta y) \qquad \textit{Forward difference}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{\Delta y} + O(\Delta y) \qquad \textit{Rearward difference}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + O(\Delta y)^2 \qquad \textit{Central difference}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + O(\Delta y)^2 \qquad \textit{Central second difference}$$

It is interesting to note that the central second difference given for example by Eq. (5.11) can be intepreted as a forward difference of the first derivatives, with rearward differences used for the first derivatives. Dropping the O notation for convenience, we have

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]_{i,j} \approx \frac{\left(\frac{\partial u}{\partial x}\right)_{i+1,j} - \left(\frac{\partial u}{\partial x}\right)_{i,j}}{\Delta x}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \left[\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right) - \left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right)\right]\frac{1}{\Delta x}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \qquad (5.12)$$

Equation (5.12) is the same difference quotient as Eq. (5.11). The same philosophy can be used to quickly generate a finite difference quotient for the mixed derivative $(\partial^2 u/\partial x\partial y)$ at grid point (i, j). For example,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad (5.13)$$

In Eq. (5.13), write the x-derivative as a central difference of the y-derivatives, and then cast the y-derivatives also in terms of central differences.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x \partial y} \approx \left[\left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y}\right) - \left(\frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y}\right)\right]\frac{1}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{1}{4\Delta x \Delta y}(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1})$$

or

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{1}{4\Delta x \Delta y}(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}) + O[(\Delta x)^2, (\Delta y)^2] \qquad (5.14)$$

Many other difference approximations can be obtained for the above derivatives ,as well as for derivatives of even higher order. The philosophy is the same. For a detailed tabulation of many forms of difference quotients, see pages 44 and 45 of Ref. [1]. What happens at a boundary? What type of differencing is possible when we have only one direction to go, namely, the direction away from the boundary? For example, consider Fig. 5.2, which illustrates a portion of the boundary, with the yaxis perpendicular to the boundary. Let grid point 1 be on the boundary, with points 2 and 3 a distance Δy and 2Δy above the boundary respectively.We wish to construct a finite difference approximation for ∂u/∂y at the boundary. It is easy to construct a forward difference as

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{u_2 - u_1}{\Delta y} + O(\Delta y) \qquad (5.15)$$

which is of first-order accuracy. However, how do we obtain a result which is of second-order accuracy? Our central difference in Eq. (5.8) fails us because it requires another point beneath the boundary, such as illustrated as point 2_ in Fig. 5.2. Point 2_is outside the domain of computation, and we generally have no information about u at this point. In the early days of CFD, many solutions attempted to sidestep this problem by assuming that u2_= u2. This is called the reflection boundary condition. In most cases it does not make physical sense, and is just as inaccurate,if not more so, than the forward difference given by Eq. (5.15).So we ask the question again, how do we find a second-order accurate finitedifference at the boundary? The answer is simple, and it illustrates another method of deriving finite-difference quotients. Assume that at the boundary u can be expressed by the polynomial

$$u = a+by+cy2 \tag{5.16}$$

Applied to the grid points in Fig. 5.2, Eq. (5.16) yields

$$u_1 = a$$
$$u_2 = a+b\Delta y+c(\Delta y)^2$$
$$u_3 = a+b(2\Delta y)+c(2\Delta y)^2$$

Solving this system for b:

$$b = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y} \tag{5.17}$$

Returning to Eq. (5.16), and differentiating:

$$\frac{\partial u}{\partial y} = b + 2cy \tag{5.18}$$

Equation (5.18), evaluated at the boundary where y = 0, yields

$$\left(\frac{\partial u}{\partial y}\right)_1 = b \tag{5.19}$$

Combining Eqs. (5.18) and (5.19), we obtain

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y} \tag{5.20}$$

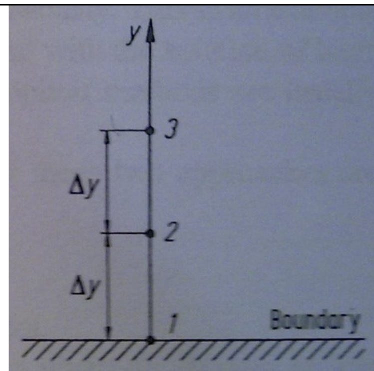
It remains to show the order-of-accuracy of Eq. (5.20). Consider a Taylor's series expansion about the point 1.

$$u(y) = u_1 + \left(\frac{\partial u}{\partial y}\right)_1 y + \left(\frac{\partial^2 u}{\partial y^2}\right)_1 \frac{y^2}{2} + \left(\frac{\partial^3 u}{\partial y^3}\right)_1 \frac{y^3}{6} + \cdots \tag{5.21}$$

Compare Eqs. (5.21) and (5.16). Our assumed polynomial expression in Eq. (5.16) is the same as using the first three terms in the Taylor's series. Hence,Eq. (5.16) is of $O(\Delta y)^3$. In forming the derivative in Eq. (5.20), we divided by Δy, which then makes Eq. (5.20) of $O(\Delta y)^2$. Thus, we can write from Eq. (5.20)

$$\left(\frac{\partial u}{\partial y}\right)_1 = \frac{-3u_1 + 4u_2 - u_3}{2\Delta y} + O(\Delta y)^2 \qquad (5.22)$$

Fig. 5.2 Grid points at a boundary



This is our desired second-order-accurate difference quotient at the boundary.Both Eqs. (5.15) and (5.22) are called one-sided differences, because they express a derivative at a point in terms of dependent variables on only one side of the point. Many other one-sided differences can be formed, with higher degrees of accuracy,using additional grid points to one side of the given point. It is not unusual to see four- and five point one-sided differences applied at a boundary.

5.3 Basic Aspects of Finite-Difference Equations

The essence of finite-difference solutions in CFD is to use the difference quotients derived in Sect. 5.2 (or others that are similar) to replace the partial derivatives in the governing flow equations, resulting in a system of algebraic difference equations for the dependent variables at each grid point. In the present section, we examine some of the basic aspects of a difference equation.Consider the following model equation, in which we assume that the dependent variable u is a function of x and t.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (5.23)$$

We choose this simple equation for convenience; at this stage in our discussions there is no advantage to be obtained by dealing with the much more complex flow equations. The basic aspects of finite-difference equations to be examined in this section can just as well be developed using Eq. (5.23). It should be noted that Eq. (5.23) is parabolic.

If we replace the time derivative in Eq. (5.23) with a forward difference, and the spatial derivative with a central difference, the result is:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{5.24}$$

In Eq. (5.24), some common notation is used for the difference of the time derivative. The index for time usually appears as a superscript in CFD, where n denotes conditions at time t,(n+1) denotes conditions at time (t+Δt), and so forth. The subscript still denotes the grid point location; for the one spatial dimension considered here, clearly we need only one index, i.

Question: What is the truncation error for the complete finite-difference equation?

Obviously, there must be a truncation error because each one of the finitedifference quotients has its own truncation error. Let us address this question. Combining Eqs. (5.23) and (5.24), and explicitly writing the truncation errors associated with the difference quotients (from Eqs. (5.4) and (5.10)), we have

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2} + \left[-\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \left(\frac{\partial^4 u}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \cdots \right] \tag{5.25}$$

Examining Eq. (5.25), on the left-hand side is the original partial differential equation, the first two terms on the right-hand side are the finite difference representation of this equation and the terms in the square brackets are the truncation error for the complete equation. Note that the truncation error for this representation is O[Δt, (Δx)²].

Does the finite-difference equation reduce to the original differential equation as the number of grid points goes to infinity, i.e. as $\Delta x \to 0$ and $\Delta t \to 0$? Examining Eq. (5.25), we note that the truncation error approaches zero, and hence the difference equation does indeed approach the original differential equation. When this is the case, the finite-difference representation of the partial differential equation is said to be consistent. The solution of Eq. (5.24) takes the form of a 'marching' solution in steps of time. (Recall from Sect. 4.3.2 that such marching solutions are a characteristic of parabolic equations.) Assume that we know the dependent variable at all x at some instant in time, say from given initial conditions. Examining Eq. (5.24), we see that it contains only one unknown, namely u_j^{n+1}. In this fashion, the dependent variable at time $(t + \Delta t)$ can be obtained explicitly from the known results at time t, i.e. u_j^{n+1} is obtained directly from the known values u^n_{j+1}, u^n_j, and u^n_{j-1}. This is an example of an explicit finite-difference solution. As a counter example, let us be daring and return to the original partial differential equation given by Eq. (5.23). This time, we write the spatial differences on the right-hand side in terms of average properties between n and (n+1), that is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[\frac{u_{i+1}^{n+1} + u_{i+1}^n - 2u_i^{n+1} - 2u_i^n + u_{i-1}^{n+1} + u_{i-1}^n}{(\Delta x)^2}\right] \qquad (5.26)$$

The differencing shown in Eq. (5.26) is called the Crank-Nicolson form. Examine Eq. (5.26) closely. The unknown u_i^{n+1} is not only expressed in terms of the known quantities at time index n, namely u^n_{i+1}, u^n_i, and u^n_{i-1}, but also in terms of unknown quantities at time index n+1, namely u^{n+1}_{i+1} and u^{n+1}_{i-1}. Hence, Eq. (5.26) applied at a given grid point i cannot by itself result in the solution for u_i^{n+1}. Rather, Eq. (5.26) must be written at all grid points, resulting in a system of algebraic equations from which the unknown u_i^{n+1} for all i can be solved simultaneously. This is an example of an implicit finite-difference solution. Because they deal with the solution of large systems of simultaneous linear algebraic equations, implicit methods are usually involved with the manipulation of large matrices. The relative major advantages and disadvantages of these two approaches are summarized as follows.

1. Explicit approach.

(a) Advantage. Relatively simple to set up and program.

(b) Disadvantage. In terms of our above example, for a given Δx, Δt must be less than some limit imposed by stability constraints. In many cases, Δt must be very small to maintain stability; this can result in long computer running times to make calculations over a given interval of t.

2. Implicit approach.

(a) Advantage. Stability can be maintained over much larger values of Δt, hence using considerably fewer time steps to make calculations over a given interval of t. This results in less computer time.

(b) Disadvantage. More complicated to set up and program.

(c) Disadvantage. Since massive matrix manipulations are usually required at each time step, the computer time per time step is much larger than in the explicit approach.

(d) Disadvantage. Since large Δt can be taken, the truncation error is larger, and the use of implicit methods to follow the exact transients (time variations of the independent variable) may not be as accurate as an explicit approach. However, for a time-dependent solution in which the steady state is the desired result, this relative time-wise inaccuracy is not important.

During the period 1969 to about 1979, the vast majority of practical CFD solutions involving 'marching' solutions (such as in the above example) employed explicit methods. Today, they are still the most straightforward methods for flow field solutions. However, many of the more sophisticated CFD applications—those requiring very closely-spaced grid points in some regions of the flow—would demand inordinately large computer running times due to the small marching steps required. This has made the advantage listed above for implicit methods very attractive, namely the ability to use large marching steps even for a very fine grid. For this reason, implicit methods are today the major focus of CFD applications.

A General Comment 5.3.1

It is clear that finite-difference solutions appear to be philosophically straightforward jus replace the partial derivatives in the governing equations with algebraic difference quotients, and grind away to obtain solutions of these algebraic equations at each grid point. However, this impression is misleading. For any given application, there is no guarantee that such calculations will be accurate, or even stable, under all conditions. Moreover, the boundary conditions for a given problem dictate the solution, and therefore the proper treatment of boundary conditions within the framework of a particular finite-difference technique is vitally important.For these reasons, finite-difference solutions of various aerodynamic flow fields are by no means routine. Indeed, much of computational fluid dynamics today is still more of an art than a science; each different problem usually requires thought and originality in its solution. However, a great deal of research in applied mathematics is now being devoted to CFD, and the next decade should see a major expansion in our understandingof the discipline, as well ,as the development of more improved efficient algorithms.[1]

5.4 Errors and an Analysis of Stability

At the end of the last section, we stated that no guarantee exists for the accuracy and stability of a system of finite-difference ,equations under all conditions. However for linear equations there is a formal way of examining the accuracy and stability and these ideas at least provide guidance for the understanding of the behaviour of the more complex non-linear system that is our governing flow equations. In this section we introduce some of these ideas, applied to simple linear equations. The material in this section is patterned somewhat after section 3–6 of the excellent new book on CFD by Dale Anderson, John Tannehill and Richard Pletcher (Ref. [1]) which should be consulted for more details. Consider a partial differential equation, such as for example Eq. (5.23). The numerical solution of this equation is influenced by two sources of error:

1. Discretization error. The difference between the exact analytical solution of the partial differential equation (for example, Eq. (5.23)) and the exact (round-off free) solution of the corresponding difference equation (for example, Eq. (5.24)).

From our previous discussion, the discretization error is simply the truncation error for the difference equation plus any errors introduced by the numerical treatment of the boundary conditions.

2. Round-off error. The numerical error introduced after a repetitive number of calculations in which the computer is constantly rounding the numbers to some significant figure.

If we let

A = analytical solution of the partial differential equation

D = exact solution of the difference equation

N = numerical solution from a real computer with finite accuracy

then ,

Discretization error = A−D

Round-off $= \varepsilon = N - D$ (5.27)

From Eq. (5.27), we can write

$$N = D + \varepsilon \qquad (5.28)$$

where again ε is the round-off error, which for the remainder of our discussion in this section, we will simply call "error" for brevity. The numerical solution N must satisfy the difference equation. Hence from Eq. (5.24),

$$\frac{D_i^{n+1} + \varepsilon_i^{n+1} - D_i^n - \varepsilon_i^n}{\Delta t} = \frac{D_{i+1}^n + \varepsilon_{i+1}^n - 2D_i^n - 2\varepsilon_i^n + D_{i-1}^n \varepsilon_{i-1}^n}{(\Delta x)^2} \qquad (5.29)$$

By definition, D is the exact solution of the difference equation, hence it exactly satisfies:

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{(\Delta x)^2} \qquad (5.30)$$

Subtracting Eq. (5.30) from (5.29),

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta x)^2} \qquad (5.31)$$

From Eq. (5.31), we see that the error ε also satisfies the difference equation.We now consider aspects of the stability of the difference equation, Eq. (5.24). If errors εi are already present at some stage of the solution of this equation (as they always are in any real computer solution), then the solution will be stable if the εi's shrink, or at best stay the same, as the solution progresses from step n to n+1; on the other hand, if the εi's grow larger during the progression of the solution from steps n to n+1, then the solution is unstable. That is, for a solution to be stable,

$$|\varepsilon_i^{n+1} / \varepsilon_i^n \leq 1 \qquad (5.32)$$

For Eq. (5.24), let us examine under what conditions Eq. (5.32) holds.Assume that the distribution of errors along the x-axis is given by a Fourier series in x, and that the time-wise variation is exponential in t, i.e.

$$\varepsilon(x,t) = e^{at} \sum_m e^{ik_m x} \qquad (5.33)$$

where km is the wave number and where the exponential factor a is a complex number. Since the difference equation is linear, when Eq. (5.33) is substituted into Eq. (5.31) the behaviour of each term of the series is the same as the series itself. Hence, let us deal with just one term of the series, and write

$$\varepsilon_m(x,t) = e^{at} e^{ik_m x} \qquad (5.34)$$

Substitute Eq. (5.34) into Eq. (5.31),

$$\frac{e^{a(t+\Delta t)}e^{ik_m x} - e^{at}e^{ik_m x}}{\Delta t} = \frac{e^{at}e^{ik_m(x+\Delta x)} - 2e^{at}e^{ik_m x} + e^{at}e^{ik_m(x-\Delta x)}}{(\Delta x)^2} \quad (5.35)$$

Divide Eq. (5.35) by $e^{at}e^{ik_m x}$.

$$\frac{e^{a\Delta t} - 1}{\Delta t} = \frac{e^{ik_m\Delta x} - 2 + e^{-ik_m\Delta x}}{(\Delta x)^2}$$

or,

$$e^{a\Delta t} = 1 + \frac{\Delta t}{(\Delta x)^2}(e^{ik_m\Delta x} + e^{-ik_m\Delta x} - 2) \quad (5.36)$$

Recalling the identity that

$$\cos(k_m\Delta x) = \frac{e^{ik_m\Delta x} + e^{-ik_m\Delta x}}{2}$$

Equation (5.36) can be written as

$$e^{a\Delta t} = 1 + \frac{2\Delta t}{(\Delta x)^2}[\cos(k_m\Delta x) - 1] \quad (5.37)$$

Recalling another trigonometric identity that

$$\sin^2[(k_m\Delta x)/2] = \frac{1 - \cos(k_m\Delta x)}{2}$$

Equation (5.37) finally becomes

$$e^{a\Delta t} = 1 - \frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2] \quad (5.38)$$

From Eq. (5.34),

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{e^{a(t+\Delta t)}e^{ik_m x}}{e^{at}e^{ik_m x}} = e^{a\Delta t} \quad (5.39)$$

Combining Eqs. (5.39), (5.38) and (5.32), we have

$$\left|\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right| = |e^{a\Delta t}| = \left|1 - \frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2]\right| \le 1 \quad (5.40)$$

Equation (5.40) must be satisfied to have a stable solution, as dictated by Eq. (5.32). In Eq. (5.40) the factor

$$\left|1-\frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2]\right| \equiv G$$

is called the amplification factor, and is denoted by G. Evaluating the inequality in Eq. (5.40), namely G ≤ 1, we have two possible situations which must hold simultaneously:

$$(1)\quad 1-\frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2] \le 1$$

Thus

$$\frac{4\Delta t}{(\Delta x)}\sin^2[(k_m\Delta x)/2] \ge 0$$

Since $\Delta t/(\Delta x)^2$ is always positive, this condition always holds.

$$(2)\quad 1-\frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2] \ge -1$$

Thus

$$\frac{4\Delta t}{(\Delta x)^2}\sin^2[(k_m\Delta x)/2] - 1 \le 1$$

For the above condition to hold,

$$\frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2} \qquad (5.41)$$

Equation (5.41) gives the stability requirement for the solution of the difference equation, Eq. (5.24), to be stable. Clearly, for a given Δx, the allowed value of Δt must be small enough to satisfy Eq. (5.41). Here is a stunning example of the limitation placed on the marching variable by stability considerations for explicit finite difference models. As long as $\Delta t/(\Delta x)^2 \le 1/2$, the error will not grow for subsequent marching steps in t, and the numerical solution will proceed in a stable manner. On the other hand, if $\Delta t/(\Delta x)^2 > 1/2$, then the error will progressively become larger, and will eventually cause the numerical marching solution to 'blow up' on the computer.The above analysis is an example of a general method called the von Neuman stability method, which is used frequently to study the stability properties of linear difference quations. Let us quickly examine the stability characteristics of another simple equation, this time a hyperbolic equation. Consider the first order wave equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad (5.42)$$

Let us replace the spatial derivative with a central difference (see Eq. (5.8)).

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x} \qquad (5.43)$$

Let us replace the time derivative with a first order difference, where u(t) is represented by an average value between grid points (i+1) and (i–1), i.e.

$$u(t) = \frac{1}{2}(u_{i+1}^{n} + u_{i-1}^{n})$$

Then

$$\frac{\partial u}{\partial t} = \frac{u_{i}^{n+1} - \frac{1}{2}(u_{i+1}^{n} + u_{i+1}^{n})}{\Delta t} \qquad (5.44)$$

Substituting Eqs. (5.43) and (5.44) into (5.42), we have

$$u_{i}^{n+1} = \frac{u_{i+1}^{n} + u_{i-1}^{n}}{2} - c\frac{\Delta t}{\Delta x}\left(\frac{u_{i+1}^{n} - u_{i-1}^{n}}{2}\right) \qquad (5.45)$$

Combining Eqs. (5.18) and (5.19), we obta The differencing used in the above equation, where Eq. (5.44) is used to represent the time derivative, is called the Lax method, after the mathematician Peter Lax who first proposed it. If we now assume an error of the form $\varepsilon_m(x, t) = e^{at}e^{ik_m t}$ as done previously, and substitute this form into Eq. (5.45), the amplification factor becomesin

$$G = \cos(k_m\Delta x) - iC\sin(k_m\Delta x) \quad (5.46)$$

where $C = c.\Delta t/\Delta x$. The stability requirement is $|e^{at}| \le 1$,which when applied to Eq. (5.46) yields

$$C = c\frac{\Delta t}{\Delta x} \le 1 \quad (5.47)$$

In Eq. (5.47), C is called the Courant number. This equation says that $\Delta t \le \Delta x/c$ for the numerical solution of Eq. (5.45) to be stable. Moreover, Eq. (5.47) is called the Courant–Friedrichs–Lewy condition, generally written as the CFL condition. It is an important stability criterion for hyperbolic equations . Let us examine the physical significance of the CFL condition. Consider the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} \quad (5.48)$$

The characteristic lines for this equation (see Sect. 4.2) are given by

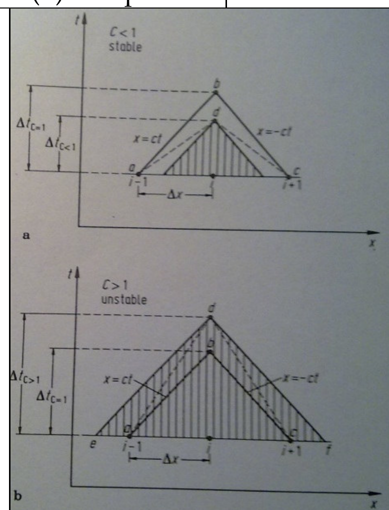
$$x = ct \quad \text{(right running)}$$

and

$$x = -ct \quad \text{(left running)}$$

and are sketched in Fig. 5.3(a) and (b). In both parts (a) and (b) of Fig. 5.3, let point b be the intersection of the right-running characteristic through grid point (i − 1) and the left-running characteristic through grid point (i+1). For Eq. (5.48), the CFL condition as given in Eq. (5.47) holds as the stability criterion. Let $\Delta t_{C=1}$ denote the value of Δt given by Eq. (5.47) when C = 1. Then $\Delta t_{C=1} = \Delta x/c$, and the intersection point b is therefore a distance $\Delta t_{C=1}$ above the x-axis, as sketched in Figs. 5.3(a) and (b). Now assume that C < 1, which is the case sketched in Fig. 5.3(a). Then from Eq. (5.47), $\Delta t_{C<1} < \Delta t_{C=1}$, as shown in Fig. 5.3(a). Let point d correspond to the grid point at point i, existing at time $(t+\Delta t_{C<1})$. Since properties at point d are calculated numerically from the difference equation using grid points (i−1) and (i+1), the numerical domain for point d is the triangle adc shown in Fig. 5.3(a). The analytical domain for point d is the shaded triangle in Fig. 5.3(a), defined by the characteristics through point d. Note that in Fig. 5.3(a) the numerical domain of point d includes the analytical domain. In contrast, consider the case shown in Fig. 5.3(b). Here, C > 1. Then, from Eq. (5.47), $\Delta t_{C>1} > \Delta t_{C=1}$, as shown in Fig. 5.3(b). Let point d

Fig. 5.3 Illustration of the physical significance of the CFL condition



in Fig. 5.3(b) correspond to the grid point i, existing at time ($t+\Delta t_{C>1}$). Since properties at point d are calculated numerically from the difference equation using grid points (i−1) and (i+1), the numerical domain for point d is the triangle adc shown in Fig. 5.3(b). The analytical domain for point d is the shaded triangle in Fig. 5.3(b),defined by the characteristics through point d. Note that in Fig. 5.3(b), the numerical domain does not include all of the analytical domain, and it is this condition which leads to unstable behaviour. Therefore, we can give the following physical interpretation of the CFL condition:

For stability, the computational domain must include all of the analytical domain.

The above considerations dealt with stability. The question of accuracy, which is sometimes quite different, can also be examined from the point of view of Fig. 5.3. Consider a stable case, as shown in Fig. 5.3(a). Note that the analytic domain of dependence for point d is the shaded triangle in Fig. 5.3(a). From our discussion in

Chap. 4, the properties at point d theoretically depend only on those points within the shaded triangle. However, note that the numerical grid points (i−1) and (i+1) are outside the domain of dependence, and hence theoretically should not influence the properties at point d. On the other hand, the numerical calculation of properties at point d takes information from grid points (i − 1) and (i + 1). This situation is exacerbated when $\Delta t_{C<1}$ is chosen to be very small, $\Delta t_{C<1} \ll \Delta t_{C=1}$. In this case, even though the calculations are stable, the results may be quite inaccurate due to the large mismatch between the domain of dependence of point d, and the location of the actual numerical data used to calculate properties at d. In light of the above discussion, we conclude that the Courant number must be equal to or less than unity for stability, $C \le 1$,but at the same time it is desirable to have C as close to unity as possible for accuracy.	

Reference

1. Anderson, D.A., Tannehill, John C. and Pletcher, Richard H., Computational Fluid Mechanics and Heat Transfer, McGraw-Hill, New York, 1984.

6 (Transformations and Grids)[8]

6.1 مدخل

If all CFD applications dealt with physical problems where a uniform, rectangular grid could be used in the physical plane, there would be no reason to alter the governing equations derived in Chap.2 we would simply apply these equations in rectangular (x,y,z,t) space, finite-difference these equations according to the difference quotients derived in Chap. 5, and calculate away, using uniform values of Δx, Δy, Δz and Δt , However ,few real problems are ever so accommodating, for exsample, assume we wish to calculate the flow over an airfoil , as sketched in Fig .6.1, where we have placed the placed the airfoil in a rectangular grid . Note the problems with this rectangular grid	

[8] Mostly from [Wendt et.al. 2009], Chapter 6 (here is the author J.D. Anderson, Jr.

:

(1) Some grid points fall inside the airfoil , where they are completely out of the flow .what values of the flow properties do we ascribe to these points?

(2) There are few , if any .grid points that fall on the surface of the airfoil . This is not good . because the airfoil surface is a vital boundary condition for the determination of the flow, and hence the airfoil surface must be clearly and strongly seen by the numerical solution.

As a result. we can conclude that the rectangular grid in Fig .6.1 is not appropriate for the solution of the flow field.In contrast, agrid that is appropriate is sketched in Fig. 6.2(a). here we see a non-uniform, curvilinear grid which is literally wrapped around the airfoil. New coordinate lines ?? and ?? = constant. This is called a boundary – fitted coordinate system , and will be discussed in detail later in this chapter. The important point is that grid points naturally fall on the airfoil surface, as shown in Fig. 6.2(a).What is equally important is that ,in the physical space shown in Fig. 6.2(a),the conventional difference quotients are difficult to use. What must be done is to transform the curvilinear grid mesh in physical space to a rectangularmesh in terms of ξ and η.This is shown in Fig. 6.2(b) which illustrates a rectangular grid in terms of ξ and η.The rectangular mesh shown in Fig. 6.2(b) is called the computational plane . There is a one-to-one correspondence between this mesh,and the curvilinear mesh in Fig. 6.2(a),called the physical plane . for example,points a,b and c in the physical plane (Fig. 6.2a) correspond to points a,b and c in the computational plane , which involves uniform $\Delta\xi$ and uniform $\Delta\eta$. The computed information is then transferred back to the physical plane. Moreover, when the governing equations are solved in the computational space, they must be expressed in terms of the variables ξ and η rather than x and y;i.e.,the governing equations must be transformed from (x, y) to (ξ, η) as the new independent variables.

The purpose of this chapter is to first describe the general transformation of the governing flow equations between the physical plane and the computational plane .

following this, various specific grids will be discussed. This material is an example of a very active area of CFD research called *grid generation*.

Fig. 6.1: Airfoil on a rectangular grid

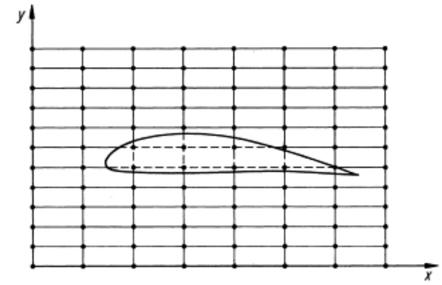
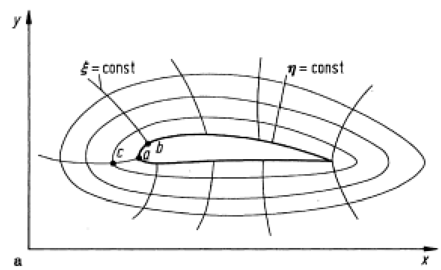
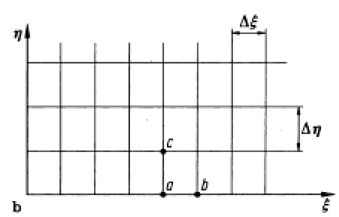


Fig. 6.2 (a) Physical plane



(b) Computational plane



General Transformation of the Equations 6.2

For simplicity , we will consider a two-dimensional unsteady flow ,with independent variables x, y and t; the results for a three-dimensional unsteady flow, with independent variables x, y ,z and t, are analogous, und simply involve more terms.
We will transform the variables in physical space(x, y, t) to a transformed space (ξ, η, τ), where

$$\xi = \xi(x, y, t) \qquad (6.1a)$$
$$\eta = \eta(x, y, t) \qquad (6.1b)$$
$$\tau = \tau(t) \qquad (6.1c)$$

In the above transformation, τ is considered a function of t only, and is frequently given by $\tau = t$ .This seems rather trivial; however , Eq.(6.1c) must be carried through the transformation in a formal manner, or else certain necessary terms will not be generated. Form the chain rule of differential calculus ,we have

$$\left(\frac{\partial}{\partial x}\right)_{y,t} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\cancelto{0}{\left(\frac{\partial \tau}{\partial x}\right)_{y,t}}$$

The subscripts in the above expression are added to emphasize what variables are being held constant in the partial differentiation. In our subsequent expression, subscripts will be dropped; however, it is always useful to keep them in your mind. Thus , we will write the above expression as

$$\boxed{\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)} \qquad (6.2)$$

Similarly,

$$\boxed{\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)} \qquad (6.3)$$

Also,

$$\left(\frac{\partial}{\partial t}\right)_{x,y} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial}{\partial \eta}\right)_{\xi,\eta}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} + \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\left(\frac{\partial \tau}{\partial t}\right)_{x,y} \qquad (6.4)$$

or,

$$\boxed{\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial t}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial t}\right) + \left(\frac{\partial}{\partial \tau}\right)\frac{d\tau}{dt}} \qquad (6.5)$$

Equations (6.2),(6.3) and (6,5) allow the derivatives with respect to x, y and t to be transformed into derivatives with respect to ξ, η and τ. The coefficients of the derivatives with respect to ξ, η and τ are called metrics, e.g. $\partial\xi/\partial x$, $\partial\xi/\partial y$, $\partial\eta/\partial x$ and $\partial\eta/\partial y$ are metric terms which can be obtained from the general transformation given by Eqs. (6.1a, b and c) .if Eqs.(6.1a ,b and c) are given as closed form analytic expressions, then the metrics can also be obtained in closed form. However, the transformation given by Eqs. (6.1a, b, and c) is frequently a purely numerical relationship, in which case the metrics can be evaluated by finite-difference quotients – typically central differences.

Examining the governing equations derived in Chap. 2, we note that the equations for viscous flow involve second derivatives. Therefore, we need a transformation for these derivatives; they can be obtained as follows. From Eq. (6.2), let

$$A = \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

Then,

$$\frac{\partial^2}{\partial x^2} = \frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial \xi}{\partial x}\right)\underbrace{\left(\frac{\partial^2}{\partial x \partial \xi}\right)}_{B} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial \eta}{\partial x}\right)\underbrace{\left(\frac{\partial^2}{\partial \eta \partial x}\right)}_{C} \qquad (6.6)$$

The mixed derivatives denoted by B and C in Eq. (6.6) can be obtained from the chain rule as follows:

$$B = \frac{\partial^2}{\partial x \partial \xi} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi}\right)$$

Recalling the chain rule given by Eq. (6.2), we have

$$B = \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right) \qquad (6.7)$$

Similarly:

$$C = \frac{\partial^2}{\partial x \partial \eta} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \eta}\right) = \left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right) \qquad (6.8)$$

Substituting B and C fro Eqs. (6.7) and (6.8) into Eq. (6.6), and rearranging the sequence of terms, we have

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right) \tag{6.9}$$

Equation (6.9) gives the second partial derivative with respect to x in terms of first, second, and mixed derivatives with respect to ξ and η, multiplied by various metric terms. Let us now continue to obtain the second partial with respect to y. From Eq. (6.3), let

$$D \equiv \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

Then,

$$\frac{\partial^2}{\partial y^2} = \frac{\partial D}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$
$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial \xi}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \xi \partial y}\right)}_{E} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial \eta}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \eta \partial y}\right)}_{F} \tag{6.10}$$

Using Eq. (6.3),

$$E = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi}\right) = \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{6.11}$$

and

$$F = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta}\right) = \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right) \tag{6.12}$$

Substituting Eqs. (6.11) and (6.12) into (6.10), we have, after rearranging the sequence of terms:

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right) \tag{6.13}$$

Equation (6.13) gives the second partial derivative with respect to y in terms of first, second, and mixed derivatives with respect to ξ and η, multiplied by various metric terms. We now continue to obtain the second partial with respect to x and y.

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial D}{\partial x} = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$
$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \xi \partial x}\right)}_{B} + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{\partial \eta}{\partial y}\right)\underbrace{\left(\frac{\partial^2}{\partial \eta \partial x}\right)}_{C} \tag{6.14}$$

Substituting Eqs. (6.7) and (6.8) for B and C

respectively into Eq. (6.14), and rearranging the sequence of terms, we have	

$$\frac{\partial^2}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)\right] \quad (6.15)$$

Equation (6.15) gives the second partial derivative with respect to x and y in terms of first, second, and mixed derivatives with respect to ξ and η, multiplied by various metric terms.

Examine all the equations given in the boxed above. They represent all that is necessary to transform the governing flow equations obtained in Chap. 2 with x, y, and t as the independent variables to ξ, η, and T as the new independent variables. Clerely, when this transformation is made, the governing equations in terms of ξ, η, and T become rather lengthy. Let us consider a simple example, namely that for inviscid, irrotational, steady, incompressible flow, for which Laplace's Equation is the governing equation.

$$\text{Laplace's Equation : } \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (6.16)$$

Transforming Eq. (6.16) from (x, y) to (ξ, η), where ξ = ξ(x, y) and η = η(x, y), we have from Eqs. (6.9) and (6.13):

$$\left(\frac{\partial^2 \phi}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial \xi \partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial \phi}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2 \phi}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial \phi}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) = 0$$

Rearranging terms, we obtain

$$\frac{\partial^2 \phi}{\partial \xi^2}\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right] + \frac{\partial^2 \phi}{\partial \eta^2}\left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right] + 2\frac{\partial^2 \phi}{\partial \xi \partial \eta}\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)\right] + \frac{\partial \phi}{\partial \xi}\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right] + \frac{\partial \phi}{\partial \eta}\left[\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right] = 0 \quad (6.17)$$

Examine Eqs. (6.16) and (6.17); the former is Laplace's equation in the physical (x, y) space, and the latter is the transformed Laplace's equation in the computational (ξ, η) space. The transformed equation clearly contains many more terms.

Once again we emphasize that Eqs. (6.1), (6.2), (6.3), (6.5), (6.9), (6.13), and (6.15) ar used to transform the governing flow equations from the physical plane (x. y space) to the computational plane (ξ , η space), and that the purpose of the transformation in most CFD applications is to transform a non-uniform grid in physical space (such as shown in Fig. 6.2a) to a uniform grid in the computational space (such as shown in Fig. 6.2b). The transformed governing partial differential equations are then finite-differenced in the computational plane, where there exists a uniform $\Delta\xi$ and a uniform $\Delta\eta$, as shown in Fig. 6.2(b). The flow-field variables are calculated at all grid points in the computational plane, such as points, a, b, and c in Fig. 6.2(b). These are the same flow-field variables which exist in the physical plane at the corresponding points a, b, and c in Fig. 6.2(a). The transformation that accomplishes all this is given in general form by Eqs. (6.1a, b, and c). Of course, to carry out a solution for a given problem, the transformation given generically by Eqs. (6.1a, b, and c) must be explicitly specified. Examples of some specific transformations will be given in subsequent sections.

6.3 Metrics and Jacobians 6.3

In Eqs. (6.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), the terms involving the geometry of the grids, such as $\partial\xi/\partial x$, $\partial\xi/\partial y$,$\partial\eta/\partial x$, $\partial\eta/\partial y$, etc., are called metrics. If the transformation, Eq. (6.1a, b and c), is given analytically, then it is possible to obtain analytic values for the metric terms.

However, in many CFD applications, the transformation, Eq. (6.1a, b and c), is given numerically, and hence the metric terms are calculated as finite differences.

Also, in many applications, the transformation may be more conveniently expressed as the inverse of Eqs. (6.1a, b), that is, we may have available the inverse transformation.

$$x = x(\xi,\eta,\tau) \qquad (6.18a)$$

$$y = y(\xi,\eta,\tau) \qquad (6.18b)$$

$$t = t(\tau) \qquad (6.18c)$$

In Eqs. (6.18a, b and c), ξ, η and τ are the *independent* variables. However, in the derivative transformations given by Eqs. (6.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), and (6.15), the metric terms $\partial\xi/\partial x$, $\partial\eta/\partial y$, etc. are partial derivatives in terms of x, y and t as the independent variables. Therefore, in order to calculate the metric terms in these equations from the inverse transformation in Eqs. (6.18a, b and c), we need to relate $\partial\xi/\partial x$, $\partial\eta/\partial y$, etc. to the inverse forms $\partial x/\partial\xi$, $\partial y/\partial\eta$, etc. These inverse forms of the metrics are the values which can be directly obtained from the inverse transformation, Eqs. (6.18a, b and c). Let us proceed to find such relations. Consider a dependent variable in the governing flow equations, such as the $x$component of velocity, u. Let $u = u(x, y)$, where from Eqs. (6.*18a* and b), $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$. The total differential of u is given by

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} \qquad (6.20)$$

Equations (6.20) and (6.21) can be viewed as two equations for the two unknowns $\partial u/\partial x$ and $\partial u/\partial y$. Solving the system of equations (6.20) and (6.21) for $\partial u/\partial x$ using Cramer's rule, we have

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}$$ 6.22	
In Eq. (6.22), the denominator determinant is identified as the *Jacobian determinant*, denoted by	
$$J \equiv \frac{\partial(x,y)}{\partial(\xi,n)} \equiv \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$	
Hence, Eq. (6.22) can be written as	
$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{J}\left[\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi}\right)\right]} \qquad (6.23)$$ 6.23	
Now let us return to Eqs. (6.20) and (6.21), and solve for $\partial u/\partial y$.	
6.24	

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial u}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial u}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}$$

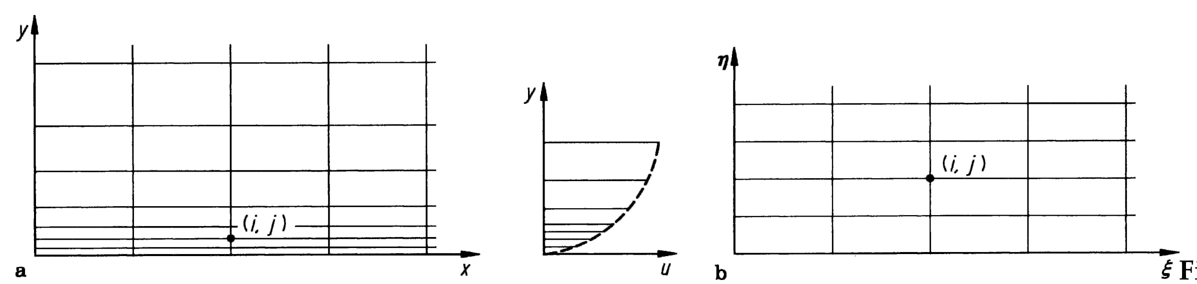
or,

$$\frac{\partial u}{\partial y} = \frac{1}{J}\left[\left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial x}{\partial \xi}\right) - \left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right)\right] \quad (6.24)$$

Examine Eqs. (6.23) and (6.24). They express the derivatives of the flow field variables in physical space in terms of the derivatives of the flowfield variables in computational space. Equations (6.23) and (6.24) accomplish the same derivative transformations as given by Eqs. (6.2) and (6.3). However, unlike Eqs. (6.2) and (6.3) where the metric terms are $\partial\xi/\partial x$, $\partial\eta/\partial y$, etc., the new Eqs. (6.23) and (6.24) involve the inverse metrics, $\partial x/\partial\xi$, $\partial y/\partial\eta$, etc. Also notice that Eqs. (6.23) and (6.24) include the Jacobian of the transformation. Therefore, whenever you have the transformation given in the form of Eqs. (6.18a, b and c), from which you can readily obtain the metrics in the form $\partial x/\partial\xi$, $\partial x/\partial\eta$, etc., the transformed governing flow equations can be expressed in terms of these inverse metrics and the Jacobian, J.A similar but more lengthy set of results can be obtained for a three-dimensional transformation from (x, y, z) to (ξ, η, ζ). Consult Ref. [1] for more details. Our discussion above has been intentionally limited to two dimensions in order to demonstratethe basic principles without cluttering the consideration with details.

6.3 Coordinate Stretching 6.4

In the remaining three sections of this chapter, we examine three types of grid transformations. The simplest is discussed here. It consists of stretching the grid in one or more coordinate directions. For example, consider the physical and computational planes shown in Fig. 6.3(a, b). Assume that we are dealing with the viscous flow over a flat surface, where the velocity varies rapidly near the surface as shown in the velocity profile sketched at the right of the physical plane (Fig. 6.3a). To calculate the details of this flow near the surface, a finely spaced grid in the *y*-direction should be used, as sketched in the physical plane. However, far away from the surface, the grid can be more coarse. Therefore, a proper grid should be one in which the coordinate lines become progressively more closely spaced as the surface is approached. On the other hand, we wish to deal with a uniform grid in the computational plane, as shown in Fig. 6.3(b). On examination, we see that the grid in the physical space is 'stretched', as if a uniform grid were drawn on a piece of rubber, and then the upper portion of the rubber were stretched upward in the *y*-direction. A simple analytical transformation which can accomplish this grid stretching is:.



Fi g. 6.3 Example of grid stretching. (**a**) Physical plane. (**b**) Computational plane

$$\xi = x \qquad (6.25a)$$

$$\eta = \ln(y+1) \qquad (6.25b)$$

The *inverse* transformation is

$$x = \xi \qquad (6.26a)$$

$$y = e^{\eta} - 1 \qquad (6.26b)$$

from which the inverse metrics are obtained as:

$$\frac{\partial x}{\partial \xi} = 1; \quad \frac{\partial x}{\partial \eta} = 0; \quad \frac{\partial y}{\partial \xi} = 0; \quad \frac{\partial y}{\partial \eta} = e^{\eta} \qquad (6.27)$$

6.22

In Eq. (6.22), the denominator determinant is identified as the *Jacobian determinant*, denoted by

Hence, Eq. (6.22) can be written as

$$\frac{\partial x}{\partial \xi} = 1; \quad \frac{\partial x}{\partial \eta} = 0; \quad \frac{\partial y}{\partial \xi} = 0; \quad \frac{\partial y}{\partial \eta} = e^{\eta} \tag{6.27}$$

Let us consider the continuity equation, given by Eq. (2.27). For steady, twodimensional flow, this is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6.28}$$

Equation (6.27) is the continuity equation written in terms of the physical plane.This equation can be formally transformed by means of the general results given by Eqs. (6.23) and (6.24), obtaining

$$\frac{1}{J}\left[\frac{\partial(\rho u)}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right) - \frac{\partial(\rho u)}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right)\right] + \frac{1}{J}\left[\frac{\partial(\rho v)}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right) - \frac{\partial(\rho v)}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right)\right] = 0 \tag{6.29}$$

Substituting into Eq. (6.29) the inverse metrics from Eq. (6.27), we have

$$e^{\eta}\frac{\partial(\rho u)}{\partial \xi} + \frac{\partial(\rho v)}{\partial \eta} = 0 \tag{6.30}$$

Equation (6.30) is the continuity equation in the computational plane.Equation (6.30) can also be obtained from the direct transformation given by Eqs. (6.25a and b). Here, the metrics are:

$$\frac{\partial \xi}{\partial x} = 1; \quad \frac{\partial \xi}{\partial y} = 0; \quad \frac{\partial \eta}{\partial x} = 0; \quad \frac{\partial \eta}{\partial y} = \frac{1}{y+1} \tag{6.31}$$

Using the transformations given by Eqs. (6.2) and (6.3), Eq. (6.28) becomes

$$\frac{\partial(\rho u)}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial(\rho u)}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial(\rho v)}{\partial \xi}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial(\rho v)}{\partial \eta}\left(\frac{\partial \eta}{\partial y}\right) = 0 \tag{6.32}$$

Substituting into Eq. (6.32) the metrics from Eq. (6.31), we have

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{1}{(y+1)}\frac{\partial(\rho v)}{\partial \eta} = 0 \tag{6.33}$$

However, from Eq. (6.26b), $y+1 = e\eta$. Therefore, Eq. (6.33) becomes

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{1}{e^{\eta}} \frac{\partial(\rho v)}{\partial \eta} = 0$$

or

$$e^{\eta} \frac{\partial(\rho u)}{\partial \xi} + \frac{\partial(\rho v)}{\partial \eta} = 0 \qquad (6.34)$$

Equation (6.34) is identical to Eq. (6.30). All that we have done here is to demonstrate how the transformed equation can be obtained from either the direct transformation or the inverse transformation ; the results are the same. An example of more complex grid stretching, in both the *x*- and *y*-directions, is given in Refs. [2, 3]. Here, the supersonic viscous flow over a blunt base is studied.The physical and computational planes are illustrated in Fig. 6.4. The streamwise stretching is accomplished through a transformation originally used by Holst [4]

$$x = \frac{\xi_0}{A}[\sinh((\xi - x_0)\beta_x) + A]$$

where

$$A = \sinh(\beta_x x_0)$$

and

$$x_0 = \frac{1}{2\beta_x} \ln\left[\frac{1 + (e^{\beta_x} - 1)\xi_0}{1 + (e^{-\beta_x} - 1)\xi_0}\right]$$

where ξ_0 is the location in the computational plane where the maximum clustering is to occur, and β_x is a constant which controls the degree of clustering at ξ_0, with larger values of B_x providing a finer grid in the clustered region. The transverse stretching is accomplished by dividing the physical plane into two sections: (1) the space directly behind the step, and (2) the space above (both in front of and behind) the step. The transformation is based on that used by Roberts [5], and is given by

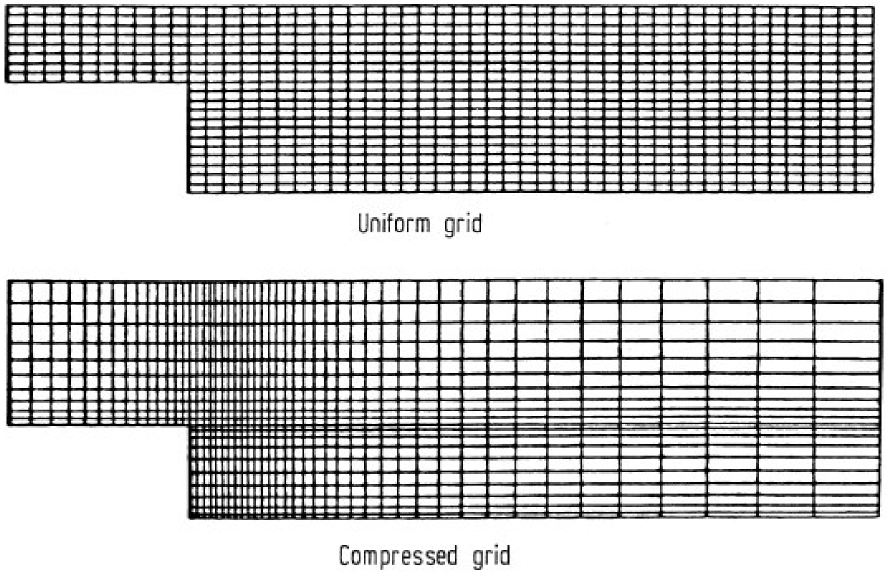


Fig. 6.4 Comparison of uniform and compressed grid

$$y = \frac{(\beta_y + 1) - (\beta_y - 1)e^{-c(\eta - 1 - \alpha)/(1-\alpha)}}{(2\alpha + 1)(1 + e^{-c(\eta - 1 - \alpha)/(1-\alpha)})}$$

where

$$c = \log\left(\frac{\beta_y + 1}{\beta_y - 1}\right)$$

and βy and α are appropriate constants, and are different for the two sections identified above. The algebraic transformations given above result in the grid stretching shown in Fig. 6.4.

6.3 Boundary-Fitted Coordinate Systems 6.5

Consider the flow through the divergent duct shown in Fig. 6.5(a). Curve *de* is the upper wall of the duct, and line *fg* is the centreline. For this flow, a simple rectangular grid in the physical plane is not appropriate, for the reasons discussed in Sect. 6.1. Instead, we draw the curvilinear grid in Fig. 6.5(a) which allows both the upper boundary *de* and the centreline *fg* to be coordinate lines, exactly fitting these boundaries. In turn, the curvilinear grid in Fig. 6.5(a) must be transformed to a rectangular grid in the computational plane, Fig. 6.5(b). This can be accomplished as follows. Let $y_s = f(x)$ be the ordinate of the upper surface *de*

in Fig. 6.5(a). Then the following transformation will result in a rectangular grid in (ξ, η) space:	

$$\xi = x$$
$$\eta = y/ys \qquad \text{where } ys = f(x)$$

The above is a simple example of a boundary fitted coordinate system. A more sophisticated example is shown in Fig. 6.6, which is an elaboration of the case illustrated in Fig. 6.2. Consider the airfoil shape given in Figure 6.6(a). A curvilinear system is wrapped around the airfoil, where one coordinate line $\eta = \eta 1$ =constant is on the airfoil surface. This is the inner boundary of the grid, designated by Γ1. The outer boundary of the grid is labelled Γ2 in Figure 6.6(a), and is given by $\eta = \eta 2$ = constant. Examining this grid, we see that it clearly fits the boundary, and hence it is a boundary-fitted coordinate system. The lines which fan out from the inner boundary Γ1 and which intersect the outer boundary Γ2 are lines of constant ξ, such as line *ef* for which $\xi = \xi 1$ = constant. (Note that in Fig. 6.6(a) the lines of constant η totally enclose the airfoil, much like elongated circles; such a grid is called an '0' type grid for airfoils. Another related curvilinear grid can have the η =constant lines trailing downstream to the right, *not* totally enclosing the airfoil (except on the inner boundary Γ1). Such a grid is called a 'C' type grid. We will see an example of a 'C' type grid shortly.)	

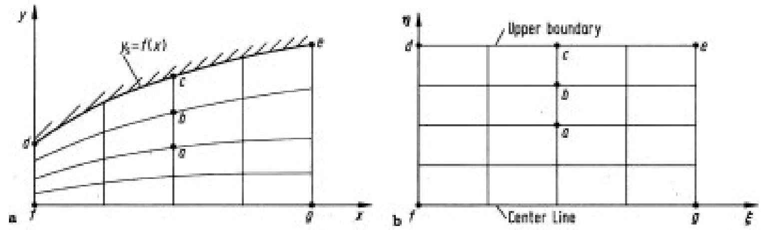
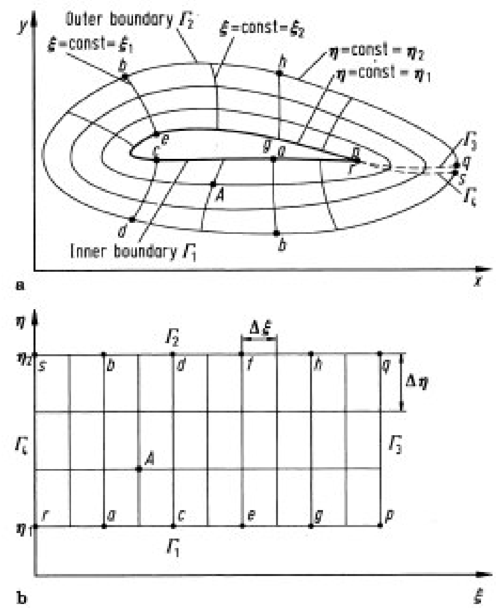


Fig. 6.5 A simple boundary-fitted coordinate system. (**a**) Physical plane. (**b**) Computational plane

6 Transformations and Grids 117

Fig. 6.6 (**a**) Physical plane. (**b**) Computational plane



Question: What transformation will cast the curvilinear grid in Fig. 6.6(a) into a uniform grid in the computational plane as sketched in Fig.6.6(b)? To answer this question, note from Fig. 6.6(a) that along the inner boundary Γ1, the physical coordinates of the body are known:

(x, y) known along Γ1

Similarly, the physical coordinates of the outer boundary Γ2 are also known, because Γ2 is simply a rather arbitrarily drawn loop around the airfoil. Once this loop Γ2 is specified, then the physical coordinates along it are known:

(x, y) known along Γ2

This hints of a boundary value problem where the boundary conditions (namely the values of x and y) are known *everywhere* along the boundary.

Recall from Sect. 4.3.3 that the solution of elliptic partial differential equations requires the specification of the boundary conditions *everywhere* along a boundary enclosing the domain. Therefore, let us consider the transformation in Fig. 6.6 to be defined by an *elliptic partial differential equation* (in contrast to an algebraic relation as illustrated in Sect. 6.4). One of the simplest elliptic equations is Laplace's equation:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{6.35a}$$

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0 \tag{6.35b}$$

where we have Dirichlet boundary conditions

$\eta = \eta_1 =$ constant on Γ_1

$\eta = \eta_2 =$ constant on Γ_2

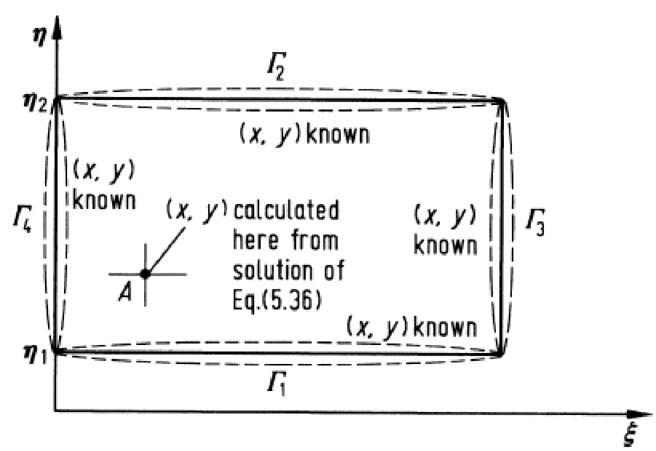
and

$\xi = \xi(x, y)$ is specified on both Γ_1 and Γ_2

It is important to keep in mind what we are doing here. The equations (6.35a and b) have *nothing* to do with the physics of the flow field. They are simply elliptic partial differential equations *which we have chosen* to relate ξ and η to *x* and *y*,and hence constitute a transformation (a one-to-one correspondence of grid points) from the physical plane to the computational plane. Because this transformation is governed by elliptic equations, it is an example of a general class of grid generation called *elliptic grid generation*. Such elliptic grid generation was first used on a practical basis by Joe Thompson at Missippi State University, and is described in detail in the pioneering paper given in Ref. [6].

Let us look more closely at the physical and computational planes shown in Fig. 6.6. In order to construct a rectangular grid in the computational plane plane (Fig. 6.6b), a cut must be made in the physical plane (Fig. 6.6a) at the trailing edge of the airfoil. This cut can be visualized as two lines superimposed on each other: the line *pq* denoted by Γ_3 represents a boundary line for the physical space above *pq*, and and the line *rs* denoted by Γ_4 represents a boundary line for the physical space below *rs*. In the physical plane, the points *p* and *r* are the same point, and the points *q* and *s* are the same point; in Fig. 6.6(a) they are slightly displaced for clarity. However, in the computational plane, these points are all different. Indeed, the grid in the computational plane is obtained by slicing the physical grid at the cut, and then 'unwrapping' the grid from the airfoil. For example, the airfoil surface in the physical plane, curve *pgecar*, becomes the lower straight line denoted by Γ1 in the computational plane. Similarly, the outer boundary *ghfdbs* becomes the upper straight line denoted by Γ2 in the computational plane. The left and right sides of the rectangle in the computational plane are formed from the cut in the physical plane; the left side is line *rs* denoted by Γ4 in Fig. 6.6(b), and the right side is line *pq* denoted by Γ3 in Fig. 6.6(b). The computational plane is sketched again in Fig. 6.7. Here we emphasize that values of (*x*, *y*) are *known along all four boundaries*, Γ1, Γ2, Γ3 and Γ4. The key aspect of the elliptic grid generation approach is that, with the given boundary conditions, Eqs. (6.35a and b) are solved for the (*x*, *y*) values which apply to *all the internal points*. An example of such an internal point is given by point *A* in Fig. 6.7, which corresponds to the same point *A* in Figs. 6.6(a) and (b). In reality, the equations to be solved are the inverse of Eqs. (6.35a and b), that is, equations obtained from Eqs. (6.35a and b) by interchanging the dependent and independent variables.The result is:

Fig. 6.7 Computational plane, illustrating the boundary conditions and an internal point



$$\alpha \frac{\partial^2 x}{\partial \xi^2} - 2\beta \frac{\partial^2 x}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 x}{\partial \eta^2} = 0 \quad (6.36a)$$

$$\alpha \frac{\partial^2 y}{\partial \xi^2} - 2\beta \frac{\partial^2 y}{\partial \xi \partial \eta} + \alpha \frac{\partial^2 y}{\partial \eta^2} = 0 \quad (6.36b)$$

where

$$\alpha = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2$$

$$\beta = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right)$$

$$\gamma = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2$$

Note in Eqs. (6.36a and b) that x and y are now expressed as the dependent variables. Returning again to Fig. 6.7, Eqs. (6.36a and b) are solved, along with the given boundary conditions for (x, y) on Γ1, Γ2, Γ3 and Γ4, to obtain the values of (x, y)which correspond to the uniformly spaced grid points in the computational (ξ, η)plane. Thus, a given grid point (ξi,ηj) in the computational plane corresponds to the *calculated* grid point (xi, yj) in physical space. The solution of Eqs. (6.36a and b) is carried out by an appropriate finite-difference solution for elliptic equations; for example, relaxation techniques are popular for such equations. Note that the above transformation, using an elliptic partial differential equation to generate the grid, does *not* involve closed-form analytic expressions; rather, it produces a set of *numbers* which locate a grid point (xi, yj) in physical space which correspond to a given grid point (ξi, ηj) in computational space. In turn, the metrics in the governing flow equations (which are solved in the computational plane), such as $\partial\xi/\partial x$, $\partial\eta/\partial y$, etc. are obtained from finite differences; central differences are frequently used for this purpose.The curvilinear, boundary-fitted coordinate system shown in Fig. 6.6(a) is simply illustrated in a qualitative sense in that figure, for purposes of instruction. An actual grid generated about an airfoil using the above elliptic grid generation approach is shown in Fig. 6.8, taken from Ref. [7]. Using Thompson's grid generation scheme(Ref. [6]), Wright ([7]) has generated a boundary-fitted coordinate system around a Miley airfoil. (The Miley airfoil is an airfoil specially designed for low Reynolds number applications by Stan Miley at Mississippi State University.) In Fig. 6.6 the white speck in the middle of the figure is the airfoil, and the grid spreads far away from the airfoil in all directions. In Ref. [7] low Reynolds number flows over airfoils were calculated by means of a time dependent finite-difference solution of the compressible Navier-Stokes equations (such time-dependent solutions are discussed in Chap. 7). The free stream is subsonic, hence the outer boundary must be placed far away from the airfoil because of the far-reaching propagation of disturbances in a subsonic flow. A detail of the grid in the near vicinity of the airfoil is shown in Fig. 6.9. Note from both Figs. 6.8 and 6.9 that the grid is a 'C' type grid, in contrast to the '0' type grid sketched in Fig. 6.6.We end this section by emphasizing again that the elliptic grid generation, with its solution of elliptic partial differential equations to obtain the internal grid points, is *completely separate* from the finite-difference solution of the governing equations. The grid is generated first, before any solution of the governing equations is attempted. The use of Laplace's equation (Eq. (6.35a and b)) to obtain this grid has nothing to do whatsoever with the physical aspects of the actual flow field. Here,Laplace's equation is simply used to generate the grid *only*.

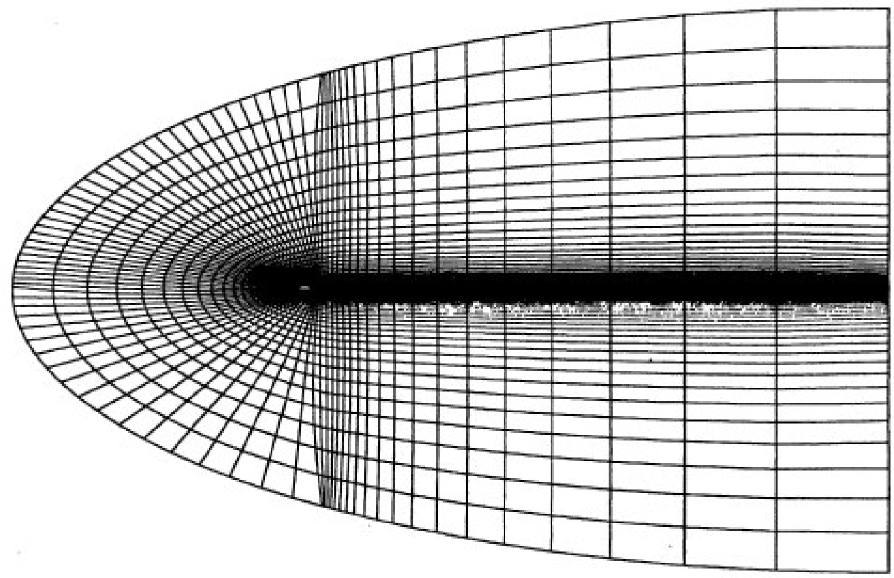


Fig. 6.8 Boundary fitted grid (from Ref. [7])

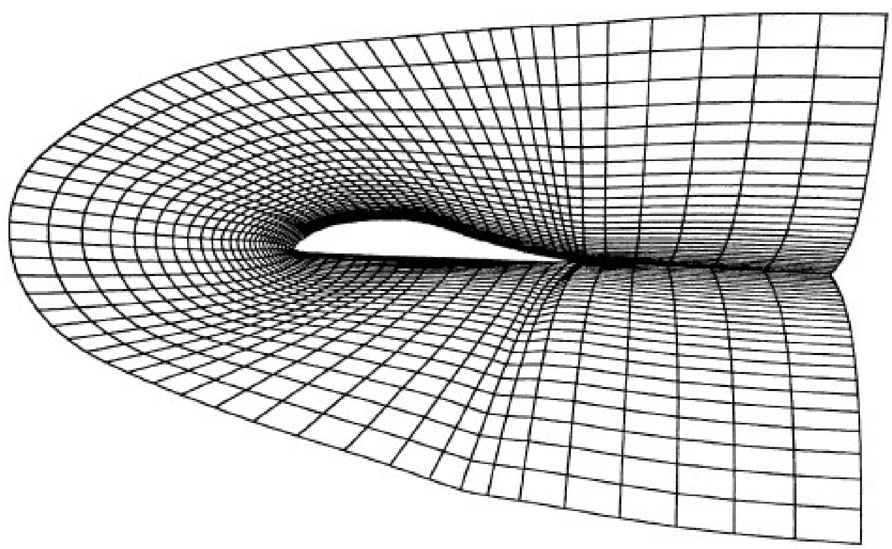


Fig. 6.9 A detail of the boundary fitted grid (from Ref. [7])

6.6 Adaptive Grids

An adaptive grid is a grid network that automatically clusters grid points in regions of high flow field gradients; it uses the solution of the flow field properties to locate the grid points	

in the physical plane. The adaptive grid evolves in steps of time in conjunction with a time dependent solution of the governing flow field equations,which computes the flow field variables in steps of time. During the course of the solution, the grid points in the physical plane *move* in such a fashion to 'adapt' to regions of large flow field gradients. Hence, the actual grid points in the physical plane are constantly in motion during the solution of the flow field, and become stationary only when the flow solution approaches a steady state. Therefore, unlike the elliptic grid generation discussed in Sect. 6.5 where the generation of the grid is completely separate from the flow field solution, an adaptive grid is intimately linked to the flow field solution, and changes as the flow field changes. The hoped-for advantages of an adaptive grid are expected because the grid points are clustered in regions where the 'action' is occurring. These advantages are: (1) increased accuracy for a fixed number of grid points, or (2), for a given accuracy, fewer grid points are needed. Adaptive grids are still very new in CFD, and whether or not these advantages are always acheived is not well established. An example of a simple adaptive grid is that used by Corda [8] for the solution of viscous supersonic flow over a rearward-facing step. Here, the transformation is expressed in the form:

$$\Delta x = \frac{B\Delta\xi}{1 + b\dfrac{\partial g}{\partial x}} \tag{6.37}$$

$$\Delta y = \frac{C\Delta\eta}{1 + c\dfrac{\partial g}{\partial y}} \tag{6.38}$$

where g is a primitive flow field variable, such as p, ϱ or T. If $g = p$, then Eqs. (6.37) and (6.38) cluster the grid points in regions of large pressure gradients; if $g = T$, the grid points cluster in regions of large temperature gradients, and so forth. In Eqs. (6.37) and (6.38), $\Delta\xi$ and $\Delta\eta$ are fixed, uniform grid spacings in the computational (ξ, η) plane, b and c are constants chosen to increase or decrease the effect of the gradient in changing the grid spacing in the physical plane, B and C are scale factors and Δx and Δy are the new grid spacings in the physical plane. Because $\partial g/\partial x$ and $\partial g/\partial y$ are changing with time during a time-dependent solution of the flow field, then clearly Δx and Δy change with time, i.e. the grid points move in the physical space. Clearly, in regions of the flow where $\partial g/\partial x$ and $\partial g/\partial y$ are large, Eqs. (6.37) and (6.38) yield small values of Δx and Δy for a given $\Delta\xi$ and $\Delta\eta$; this is the mechanism which clusters the grid points.In dealing with an adaptive grid, the computational plane consists of fixed points in the (ξ, η) space; these points are fixed in time, i.e. they do *not* move in the computational space. Moreover, $\Delta\xi$ is uniform, and $\Delta\eta$ is uniform. Hence, the computational plane is the same as we have discussed in previous sections.

The governing flow equations are solved in the computational plane, where the x, y and t derivatives are transformed according to Eqs. (6.2), (6.3) and (6.5). In particular, examine the transformation given by Eq. (6.5) for the time derivative. In the case of stretched or boundary-fitted grids as discussed in Sects. 6.4 and 6.5 respectively, the metrics $\partial\xi/\partial t$ and $\partial\eta/\partial t$ were zero, and Eq. (6.5) yields $\partial/\partial t = \partial/\partial\tau$. However, for an adaptive grid,

$$\frac{\partial \xi}{\partial t} \equiv \left(\frac{\partial \xi}{\partial t}\right)_{x,y}$$

and

$$\frac{\partial \eta}{\partial t} \equiv \left(\frac{\partial \eta}{\partial t}\right)_{x,y}$$

are finite. Why? Because, although the grid points are fixed in the computational plane, the grid points in the physical plane are moving with time. The physical meaning of $(\partial\xi/\partial t)_{x,y}$ is the time rate of change of ξ at a *fixed* (x, y) location in the physical plane. Similarly, the physical meaning of $(\partial\eta/\partial t)_{x,y}$ is the time rate of change of η at a *fixed* (x, y) location in the physical plane. Imagine that you have your eyes locked to a fixed (x, y) point in the physical plane. As a function of time, the values of ξ and η associated with this *fixed* (x, y) point will change. This is why $\partial\xi/\partial t$ and $\partial\eta/\partial t$ are finite. In turn, when dealing with the transformed flow equations in the computational plane, all three terms on the right-hand side of Eq. (6.5) are finite, and must be included in the transformed equations. In this fashion, the time metrics ∂ξ/∂t and ∂η/∂t automatically take into account the movement of the adaptive grid during the solution of the governing flow equations.

The values of the time metrics in the form shown in Eq. (6.5) are a bit cumbersome to evaluate; on the other hand, the related time metrics

$$\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} \quad \text{and} \quad \left(\frac{\partial y}{\partial t}\right)_{\xi,\eta}$$

are much easier to evaluate, because they come from

$$\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} \approx \frac{\Delta x}{\Delta t} \tag{6.39}$$

and

$$\left(\frac{\partial y}{\partial t}\right)_{\xi,\eta} \approx \frac{\Delta y}{\Delta t} \tag{6.40}$$

where Δx and Δy are obtained directly from the transformation given in Eqs. (6.37) and (6.38) respectively. Let us find the relationship between these two sets of time metrics. Consider

$$x = x(\xi, \eta, \tau)$$

Hence

$$\mathrm{d}x = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} \mathrm{d}\xi + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \mathrm{d}\eta + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} \mathrm{d}\tau$$

From this result, we write

$$\cancel{\left(\frac{\partial x}{\partial t}\right)_{x,y}}^{\,0} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} \left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \left(\frac{\partial \eta}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} \cancel{\left(\frac{\partial \tau}{\partial t}\right)_{x,y}}^{\,1}$$

or

$$-\left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} \left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \left(\frac{\partial \eta}{\partial t}\right)_{x,y} \tag{6.41}$$

Note that we are carrying the subscripts on the partial derivatives to avoid any confusion over what variables are held constant. Now consider

$$y = y(\xi, \eta, \tau)$$

Hence:

$$\mathrm{d}y = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau} \mathrm{d}\xi + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} \mathrm{d}\eta + \left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} \mathrm{d}\tau$$

Thus, from this result we write

$$\cancelto{0}{\left(\frac{\partial y}{\partial t}\right)_{x,y}} = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} + \left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta}\cancelto{1}{\left(\frac{\partial \tau}{\partial t}\right)_{x,y}}$$

or

$$-\left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} \tag{6.42}$$

Solve Eqs. (6.41) and (6.42) for $\left(\frac{\partial \xi}{\partial t}\right)_{x,y}$

$$\left(\frac{\partial \xi}{\partial t}\right)_{x,y} = \frac{\begin{vmatrix} -\left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} & \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \\ -\left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} & \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} \end{vmatrix}}{\begin{vmatrix} \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} & \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} \\ \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau} & \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} \end{vmatrix}}$$

Recognizing that $\tau = t$, and that the denominator is the Jacobian J, the above equation becomes (dropping subscripts)

$$\frac{\partial \xi}{\partial t} = \frac{1}{J}\left[-\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \eta}\right)\right] \tag{6.43}$$

Solving Eqs. (6.41) and (6.42) for $\left(\frac{\partial \eta}{\partial t}\right)_{x,y}$, , we find a likewise fashion that

$$\frac{\partial \eta}{\partial t} = \frac{1}{J}\left[\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \xi}\right) - \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \xi}\right)\right] \tag{6.44}$$

Let us recapitulate. For an adaptive grid, the governing flow equations, when transformed for solution in the computational (ξ, η) plane, must contain all the terms in the time transformation given by Eq. (6.5). The time metrics, $\partial\xi/\partial t$ and $\partial\eta/\partial t$, in Eq. (6.5) can in turn be expressed in terms of $\partial x/\partial t$ and $\partial y/\partial t$ through Eqs. (6.43) and (6.44). These new time metrics can in turn be readily calculated from Eqs. (6.39) and (6.40), where Δx and Δy are given by the basic transformation in Eqs. (6.37) and (6.38). An example of an adapted grid for the supersonic viscous flow over a rearward facing step is given in Fig. 6.10, taken from the work of Corda [8]. Flow is from left to right. Note that the grid points cluster around the expansion wave from the top corner of the step, and around the reattachment shock wave downstream of the step.It is interesting to note that the adapted grid itself is a type of 'flow field visualization method' that helps to identify the location of waves and other gradients in the flow. As a final note, there are many different approaches for the generation of adaptive grids. The above discussion is just one; it is based on ideas presented by Dwyer et al. in Ref. [9]. For a more complete discussion on adaptive grids, as well as grid generation in general, see Ref. [1].

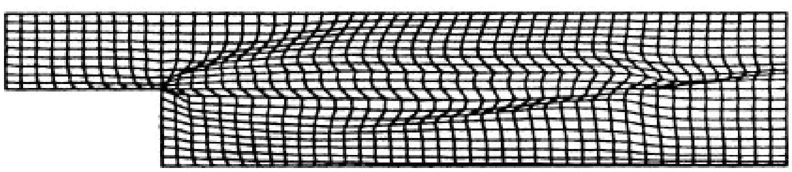


Fig. 6.10 Adapted grid for the rearward-facing step problem (from Corda, Ref. [8])

References

1. Anderson, D.A., Tannehill, John C. and Pletcher, Richard H., *Computational Fluid Mechanics and Heat Transfer*, McGraw-Hill, New York, 1984.
2. Sullins, G.A., Anderson, J.D., Jr. and Drummond, J.P., 'Numerical Investigation of Supersonic Base Flow with Parallel Injection,' AIAA Paper No. 82–1001.
3. Sullins, G.A., Numerical Investigation of Supersonic Base Flow with Tangential Injection, M.S. Thesis, Department of Aerospace Engineering, University of Maryland, 1981.
4. Holst, T.L., 'Numerical Solution of Axisymmetric Boattail Fields with Plume Simulators,' AIAA Paper No. 77–224, 1977.
5. Roberts, B.O., 'Computational Meshes for Boundary Layer Problems,' *Lecture Notes in Physics*, Springer-Verlag, New York, 1971, pp. 171–177.
6. Thompson, J.F., Thames, F.C. and Mastin, C.W., 'Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate Systems for Fields Containing Any Number of Arbitrary Two-Dimensional Bodies,' *Journal of Computational Physics*, Vol. 15, pp. 299–319, 1974.
7. Wright, Andrew F., A Numerical Investigation of Low Reynolds Number Flow Over an Airfoil, M.S. Thesis, Department of Aerospace Engineering, University of Maryland, 1982.

8. Corda, Stephen, Numerical Investigation of the Laminar, Supersonic Flow over a Rearward-*Facing Step Using an Adaptive Grid Scheme*, M.S. Thesis, Department of Aerospace Engineering, University of Maryland, 1982.
9. Dwyer, H.A., Kee, R.J. and Sanders, B.R., 'An Adaptive Grid Method for Problems in Fluid Mechanics and Heat Transfer,' AIAA Paper No. 79–1464, 1979.

Chapter 7 (Explicit Finite Difference Methods: Some Selected Applications to Inviscid and Viscous Flows) 7

7.1 Introduction

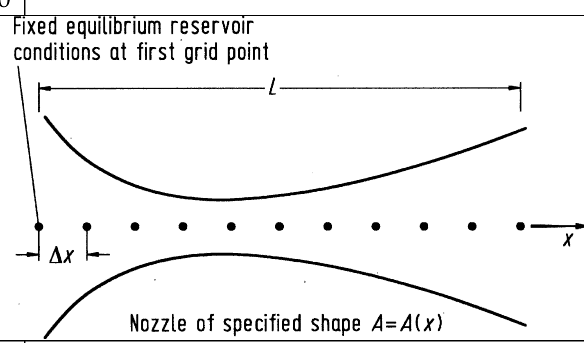
In this chapter we round-out our introductory treatment of computational fluid dynamics by discussing some applications of explicit finite difference methods to selected examples for inviscid and viscous flows. These examples have one thing in common—they are results obtained by either the present author and/or some of his graduate students over the past few years. This is not meant to be chauvinistic; rather this choice is intentionally made to illustrate what can be done by uninitiated students who are new to the ideas of CFD. These examples demonstrate the power and beauty of CFD in the hands of students much like yourselves who may have little or no experience in the field. Moreover, in all cases the applications are carried out with computer programs designed and written completely by each student. This is following the author's educational philosophy that each student should have the experience of starting with paper and pencil, writing down the governing equations, developing the appropriate numerical solution of these equations, writing the FORTRAN program, punching the program into the computer, and then going through all the trials and tribulations of making the program work properly. This is an important aspect of CFD education. No established computer programs ('canned' programs) are used; everything is 'home-grown', with the exception of standard graphics packages which are used to plot the results. Therefore, by examining these examples, you should obtain a reasonable feeling for what you can expect to accomplish when youfirst jump into the world of CFD applications.Before we discuss some examples, it is important to describe the mechanism of explicit finite-difference calculations. The distinction between explicit and implicit approaches was made in Sect. 5.3, which should be reviewed before progressing further in this chapter. In the next few sections, we will describe two rather straightforward and popular explicit methods. The treatment and application of implicit methods is given by other lectures in this course, and hence will not be discussed here.

Finally, the examples discussed in this chapter all incorporate the time-dependent method, i.e. forward marching in steps of time. The historic break-through made by this method in the 1960s is discussed in Chap. 1. The vast majority of time dependent solutions have as their objective the solution of a steady-state flow field which is approached by the solution at large times; here, the time-dependent mechanism is simply a means towards achieving that end. In other applications, the timedependent method is used to calculate the actual transients in an unsteady flow of interest. Examples of both are given here. We note, however, that although the following sections deal with marching forward in time, the same techniques are easily applied to a steady flow calculation where spatial marching is done along some coordinate axis. We have seen in Chap. 4 that such forward marching (in time or space) is appropriate when the governing equations are hyperbolic or parabolic.

7.2 The Lax–Wendroff Method

Let us describe this method by considering a simple gas-dynamic problem, namely the subsonic–supersonic isentropic flow through a convergent–divergent nozzle, as sketched in Fig. 7.1. Here, a nozzle of specified area distribution, $A = A(x)$, is given, and the reservoir conditions are known. Let us consider a quasi-one-dimensional solution where the flow field variables are functions of x (in the steady state). For a calorically perfect gas, the solution of this flow is classical, and can be found in any compressible **flow** text book (see for example Refs. [1,2]).We use this example here only because it is an excellent vehicle for introducing and describing the time dependent finite-difference philosophy.The nozzle is divided into a number of grid points in the x-direction as shown in Fig. 7.1; the spacing between adjacent grid points is Δx. Now assume values of the flow field variables at all grid points, and consider this rather arbitrarily assumed flow as an initial condition at time t = 0. In general,these assumed values will not be the exact stead state results; indeed, the exact steady-state results are what we are trying to calculate. Consider a grid point, say point i. Let gi denote a flow field variable at this point (gi might be pressure, density, velocity, etc.). This variable gi will be a function of time; however, we know gi at time t = 0, i.e. we know gi(0) because we have assumed values for all the flow field variables at all the grid points at the initial time t = 0

Fig. 7.1 Flow through a convergent-divergent nozzle



We now calculate a new value of gi at time t +Δt; starting from the initial conditions, the first new time is t+Δt = 0+Δt. Here, Δt is a small increment in time to be discussed later. The new value of gi, i.e. gi(t+Δt), is obtained from a Taylor’s series expansion in time as

$$g_i(t+\Delta t) = g_i(t) + \left(\frac{\partial g}{\partial t}\right)_i \Delta t + \left(\frac{\partial^2 g}{\partial t^2}\right)_i \frac{(\Delta t)^2}{2} + \cdots$$

or, using the standard notation of time as a superscript,

$$g_i^{t+\Delta t} = g_i^t + \left(\frac{\partial g}{\partial t}\right)_i^t \Delta t + \left(\frac{\partial^2 g}{\partial t^2}\right)_i^t \frac{(\Delta t)}{2} + \cdots \quad (7.1)$$

Here $g_i^{t+\Delta t}$ is the value of g at grid point i and at time t +Δt; $(\partial g/\partial t)_i^t$ is the first partial of g evaluated at grid point i at time t, etc. In Eq. (7.1), g_i^t is known and Δt is specified. Therefore, we can use Eq. (7.1) to calculate $g_i^{t+\Delta t}$ if we have numbers for the derivatives $(\partial g/\partial t)_i^{t+\Delta t}$ and $(\partial 2g/\partial t2)_i^{t+\Delta t}$. The numbers for the derivatives are obtained from the physics of the flow as embodied in the governing flow equations. (Note that Eq. (7.1) is simply mathematics, and by itself is certainly not sufficient to solve the problem.) The governing flow equations for the quasi-one-dimensional flow through a nozzle are (14):

$$\text{Continuity}: \quad \frac{\partial \rho}{\partial t} = -\frac{1}{A}\frac{\partial(\rho u A)}{\partial x} \quad (7.2)$$

$$\text{Momentum}: \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x} + \rho u \frac{\partial u}{\partial x}\right) \quad (7.3)$$

$$\text{Energy}: \quad \frac{\partial e}{\partial t} = -\frac{1}{\rho}\left[p\frac{\partial u}{\partial x} + pu\frac{\partial(1nA)}{\partial x} + \rho u \frac{\partial e}{\partial x}\right] \quad (7.4)$$

Note that Eqs. (7.2), (7.3) and (7.4) are written with the time derivatives on the left-hand side, and spatial derivatives on the right-hand side. For the moment, let us calculate density, i.e. g ≡ ϱ, and let us consider just the continuity equation, Eq. (7.2).Expanding the right-hand side of Eq. (7.2), we obtain

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\rho u \frac{\partial A}{\partial x} - u\frac{\partial \rho}{\partial x} - \rho \frac{\partial u}{\partial x} \quad (7.5)$$

At time t = 0, the flow field variables are assumed; hence we can replace the spatial derivatives with central differences:

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\frac{1}{A}\rho_i^t u_i^t \left(\frac{A_{i+1} - A_{i-1}}{2\Delta x}\right) - u_i^t\left(\frac{\rho_{i+1}^t - \rho_{i-1}^t}{2\Delta x}\right) - \rho_i^t\left(\frac{u_{i+1}^t - u_{i-1}^t}{2\Delta x}\right) \quad (7.6)$$

Equation (7.6) gives us a number for $(\partial \varrho/\partial t)_i^t$, which is inserted into Eq. (7.1).However, to complete Eq. (7.1), we need a number for the second partial also, namely $(\partial^2\rho/\partial t^2)^t_i$. To obtain this, differentiate the continuity equation, Eq. (7.5), with respect to time:

$$\frac{\partial^2\rho}{\partial t^2} = -\frac{1}{A}\left[\frac{\partial A}{\partial x}\left(\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}\right)\right] - u\frac{\partial^2\rho}{\partial x\partial t} - \left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial u}{\partial t}\right) - \rho\frac{\partial^2 u}{\partial x\partial t} - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \rho}{\partial t}\right) \quad (7.7)$$

Also, differentiate the continuity equation, Eq. (7.5), with respect to x:

$$\frac{\partial^2\rho}{\partial t\partial x} = -\frac{1}{A}\left[\rho u\frac{\partial^2 A}{\partial x^2} + \left(\frac{\partial A}{\partial x}\right)\left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right)\right] - u\frac{\partial^2\rho}{\partial x^2} - \left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) - \rho\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \rho}{\partial x}\right) \quad (7.8)$$

The procedure now works as follows:

(1) In Eq. (7.8), replace all derivatives on the right-hand side with central differences, such as

$$\frac{\partial u}{\partial x} = \frac{u^t_{i+1} - u^t_{i-1}}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u^t_{i+1} - 2u^t_i + u^t_{i-1}}{(\Delta x)^2}$$

etc.

This now provides a number for $(\partial^2\rho/\partial t\partial x)^t_i$ from Eq. (7.8).

(2) Insert this number for $(\partial^2\rho/\partial t\partial x)^t_i$ into Eq.(7.7). Also in Eq. (7.7), numbers for $\partial u/\partial t$ and $\partial^2 u/\partial x\partial t$ are obtained from a treatment of the momentum equation,Eq. (7.3), in a manner exactly the same as the continuity equation was treated above. The details will not be given here. In Eq. (7.7), a number for $(\partial\rho/\partial t)$ is already available, namely from Eq. (7.6). The net result is that we now have a number for $(\partial^2\rho/\partial t^2)^t_i$, obtained from Eq. (7.7).

(3) Insert this number for $(\partial^2\rho/\partial t^2)^t_i$ into Eq. (7.1) remembering that $g \equiv \rho$ for this case.

(4) Insert the number for $(\partial\rho/\partial t)^t_i$, obtained from Eq. (7.6), into Eq. (7.1).

(5) Every quantity on the right-hand side of Eq. (7.1) is now known. This allows the density $\rho_i^{t+\Delta t}$ to be calculated from Eq. (7.1). This is indeed what we wanted.We now have the density at grid point i at the next step in time, t+Δt.

(6) Perform the above procedure at every grid point to obtain $\rho(t+\Delta t)$ everywhere throughout the nozzle.

(7) Perform the above procedure on the momentum and energy equations to obtain u(t + Δt) and e(t + Δt) everywhere throughout the nozzle. We now have the complete flowfield at time (t + Δt), obtained from theknown flowfield at time t. (Recall that the process is started at t = 0 with the assumed initial conditions.)

(8) Repeat the above process for a large number of time steps. At each time step, the flow properties at all grid points will change from one time to the next.However, at large times, these changes become very small, and a steady-state is approached. This steady-state is the desired result, and the time-dependent technique is simply a means to that end.

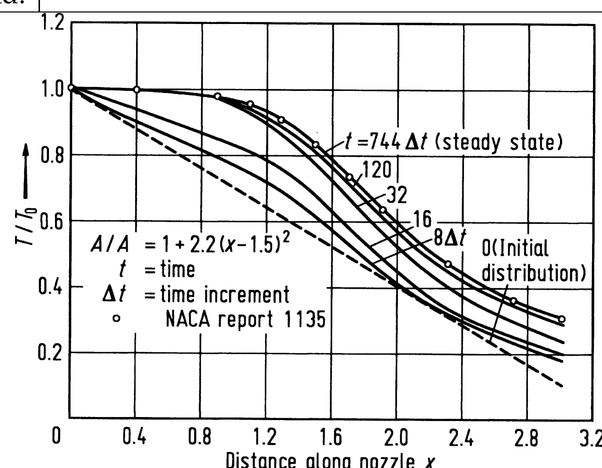
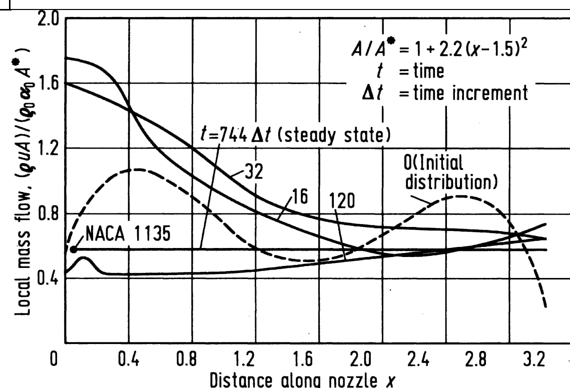


Fig. 7.2 Transient and final steady-state temperature distributions for a calorically perfect gas obtained from the present time dependent analysis, $\gamma = 1.4$

The behaviour of this type of solution is illustrated in Figs. 7.2 and 7.3. In Fig. 7.2, the temperature distribution through a given nozzle is shown. The dashed line labelled t = 0 is the initially assumed values for T throughout the nozzle. The curve above it labelled 8Δt is the temperature distribution after eight time steps following the above procedure. The curves labeled 16Δt and 32Δt are similar results after 16 and 32 time steps respectively. Note that the temperature distribution has rapidly changed from the assumed initial distribution at t = 0. At later times, the changes become smaller; note that the curve labelled 120Δt is not too different from that for 32Δt. Finally, after 744 time steps, the changes are so small that the temperature distribution is essentially at a steady state. This steady state is the desired solution. Note that the numerically-obtained steady state agrees virtually perfectly with the classical results, as can be obtained from Refs. [1, 3], and from Ref. [4].Fig. 7.3 illustrates the variation of mass flow, $\dot{m}$, through the nozzle. The dashed line is the $\dot{m}$ consistent with the assumed initial conditions at t = 0. The curves labeled 16Δt and 32Δt graphically demonstrate the wild variations in $\dot{m}$ at early times.

Fig. 7.3 Transient and final steady-state mass-flow distributions for a calorically perfect gas obtained from the present time-dependent analysis, $\gamma = 1.4$



However, after 120 time steps $\dot{m}$ has become more stable, and after 744 time steps has reached a steady state. This steady state distribution for $\dot{m}$ is a straight, horizontal line, as it should be for steady flow, where $\dot{m}$ = constant through the nozzle.Moreover, it is the correct value of mass flow, as compared to results from Ref. [4]. The method described above, utilizing Eq. (7.1), which is the first three terms of a Taylor's series expansion and where both the first and second partial derivatives in Eq. (7.1) are found by finite-differencing the spatial derivatives in the governing flow equations with central differences, is called the Lax-Wendroff method. Note that the method is of second-order accuracy, from Eq. (7.1). This method was employed with much success in the late 1960s until a more straight-forward version of the same idea was introduced by MacCormack in 1969. This is the subject of the next section. For more details about the Lax-Wendroff method as applied to the nozzle problem, see Refs. [5, 6].

7.3 MacCormack's Method

MacCormack's method, first introduced in 1969 (see Ref. [7]), has been the most popular explicit finite-difference method for solving fluid flows. It is closely related to the Lax-Wendroff method, but is easier to apply. Let us use the same nozzle problem discussed in Sect. 7.2 to illustrate MacCormack's method in the present section. MacCormack's method, like the Lax-Wendroff method, is based on a Taylor's series expansion in time. Once again, as in Sect. 7.2, let us consider the density at grid point i.

$$\rho_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_{ave} \Delta t \qquad (7.9)$$

Equation (7.9) is a truncated Taylor's series that looks first-order accurate; however, $(\partial\rho/\partial t)_{ave}$ is an average time derivative taken between time t and t +Δt. This derivative is evaluated in such a fashion that the calculation of $\rho_i^{t+\Delta t}$ from Eq. (7.9) becomes second-order accurate. The average time derivative in Eq. (7.9) is evaluated from a predictor-corrector philosophy as follows.Predictor step.We repeat the continuity equation, Eq. (7.5), below:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\rho u \frac{\partial A}{\partial x} - u\frac{\partial \rho}{\partial x} - \rho \frac{\partial u}{\partial x} \qquad (7.5 \text{ repeated})$$

In Eq. (7.5), calculate the spatial derivatives from the known flow field values at time t using forward differences. That is, from Eq. (7.5),

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\frac{1}{A}\left[\rho_i^t u_i^t \left(\frac{A_{i+1} - A_i}{\Delta x}\right)\right] - u_i^t \left(\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}\right) - \rho_i^t \left(\frac{u_{i+1}^t - u_i^t}{\Delta x}\right) \qquad (7.10)$$

Obtain a predicted value of density, $\bar{\rho}_i^{t+\Delta t}$, from the first two terms of a Taylor's series, as follows

$$\bar{\rho}_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^t \Delta t \qquad (7.11)$$

In Eq. (7.11), ρ_i^t is known, and $(\partial\rho/\partial t)_i^t$ is a known number from Eq. (7.10); hence, $\bar{\rho}_i^{t+\Delta t}$ is readily obtained. In a similar fashion, from the momentum and energy equations, predicted values of the other flow variables such as $\bar{u}_i^{t+\Delta t}$, $\bar{e}_i^{t+\Delta t}$, etc. areobtained. Corrector step Here, we first obtain a predicted value of the time derivative, $(\partial\rho/\partial t)_i^{t+\Delta t}$, by substituting the predicted values of $u_i^{t+\Delta t}$, $\rho_i^{t+\Delta t}$, etc. into Eq. 7.5, using rearward differences.

$$\overline{\left(\frac{\partial\rho}{\partial t}\right)}_i^{t+\Delta t} = -\frac{1}{A}\bar{\rho}_i^{t+\Delta t}\bar{u}_i^{t+\Delta t}\left(\frac{A_i - A_{i-1}}{\Delta x}\right) - \bar{u}_i^{t+\Delta t}\left(\frac{\bar{\rho}_i^{t+\Delta t} - \bar{\rho}_{i-1}^{t+\Delta t}}{\Delta x}\right) - \bar{\rho}_i^{t+\Delta t}\left(\frac{\bar{u}_i^{t+\Delta t} - \bar{u}_{i-1}^{t+\Delta t}}{\Delta x}\right) \qquad (7.12)$$

Now calculate the average time derivative as the arithmetic mean between Eqs. (7.10) and (7.12), i.e.

$$\left(\frac{\partial\rho}{\partial t}\right)_{ave} = \frac{1}{2}\left[\left(\frac{\partial\rho}{\partial t}\right)_i^t + \overline{\left(\frac{\partial\rho}{\partial t}\right)}_i^{t+\Delta t}\right] \qquad (7.13)$$

where numbers for the two terms on the right-hand side of Eq. (7.13) come from Eqs (7.10)and (7.12) respectively. Finally, we obtain the corrected value of $\rho_i^{t+\Delta t}$ from Eq. (7.9), repeated below:

$$\rho_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial\rho}{\partial t}\right)_{ave}\Delta t \qquad \text{(7.9 repeated)}$$

The above predictor–corrector approach is carried out for all grid points throughout the nozzle, and is applied simultaneously to the momentum and energy equations in order to generate $u_i^{t+\Delta t}$ and $e_i^{t+\Delta t}$.In this fashion, the flow field through the entire nozzle at time t +Δt is calculated. This is repeated for a large number of time steps until the steady state is achieved, just as in the case of the Lax Wendroff method described in Sect. 7.2. MacCormack's technique as described above, because a two-step predictor–corrector sequence is used with forward differences on the predictor and rearward differences on the corrector, is a second-order accurate method. Therefore, it has the same accuracy as the Lax-Wendroff method described in Sect. 7.2. However, the MacCormack method is much easier to apply, because there is no need to evaluate the second time derivatives as was the case for the Lax-Wendroff method. To see this more clearly, recall Eqs. (7.7) and (7.8), which are required for the Lax-Wendroff method. These equations represent a large number of additional calculations. Moreover, for a more

complex fluid dynamic problem, the differentiation of the continuity, momentum and energy equations to obtain the second derivatives, first with respect to time, and then the mixed derivatives with respect to time and space, can be very tedious, and provides an extra source for human error. MacCormack's method does not require such second derivatives, and hence does not deal with equations such as Eqs. (7.7) and (7.8). A few comments are made with regard to the specific application to the quasione dimensional nozzle flow shown in Fig. 7.1. At the inflow boundary (the first grid point at the left), the values of p, T and ϱ are fixed, independent of time, and are assumed to be reservoir values. The inflow velocity, which is a very small subsonic value, is calculated from linear extrapolation using the adjacent internal points, or it can be evaluated from the momentum equation applied at the first grid point using one-sided differences. At the outflow boundary (the last grid point at the right in Fig. 7.1), all the dependent variables are obtained from linear extrapolation from the adjacent internal points, or by applying the governing equations at this point, using one-sided differences. Finally, we note that results obtained from the Lax–Wendroff method and from the MacCormack method are virtually identical. For example, these two methods are compared for a vibrationally relaxing, high temperature, non-equilibrium nozzle flow in Ref. [8]; there is no difference between the two sets of results.	

7.4 Stability Criterion

Examine Eq. (7.1), which is vital to the Lax–Wendroff method. Note that it requires the specification of a time increment, Δt. Examine Eqs. (7.9) and (7.11), which are vital to the MacCormack method. They too require the specification of a time increment ,Δt. For explicit methods, the value of Δt cannot be arbitrary, rather it must be less than some maximum value allowable for stability. The time-dependent applications described in Sects. 7.2 and 7.3 are dealing with governing	

flow equations which are hyperbolic with respect to time. Recall our discussion in Sect. 5.4 dealing with the stability criteria for such equations. There, it was stated that Δt must obey the Courant–Friedrichs–Lewy criterion— the so-called CFL criterion. This is embodied in Eq. (5.47), which was derived from the simple model equation given by Eq. (5.42). This is the linear wave equation, where c is the wave propagation speed.If the wave were propagating through a gas which already has a velocity u, then the wave will travel at the velocity (u + c) relative to the stationary surroundings. For such a case, Eq. (5.47) becomes

$$\Delta t = C\left(\frac{\Delta x}{u+c}\right); \quad C \leq 1 \tag{7.14}$$

where C is the Courant number, and c is the speed of sound, c = (∂p/∂ϱ)s. Eq. (7.14) is the appropriate CFL criterion for the one dimensional, explicit solutions of nozzle flows discussed in Sects. 7.2 and 7.3. The CFL criterion given by Eq. (7.14) says physically that the explicit time step must be no greater than the time required for asound wave to propagate from one grid point to the next. This author's experience has been that C should be as close to unity as possible, but depending upon the actual application, themaximumallowable value ofC for stability in explicit timedependent finite difference calculations can vary from approximately 0.5–1.0.Keep in mind that the stability criteria exemplified by Eqs. (5.47) and (7.14) are based on analysis of linear equations. On the other hand, the governing equations for a general fluid flow are highly non- linear . Therefore,wewould not expect theCFLcriteria to apply exactly to such cases; instead, it provides a reasonable estimate of Δt for a given non-linear problem, and as a result the value of the Courant number in Eq. (7.14) can be viewed as an adjustable parameter to compensate for such non-linearities. Return for a moment to the nozzle flow application discussed in Sects. 7.2 and 7.3. Here, at any given time t, Eq. (7.14) is evaluated at each grid point throughout the flow. Because u and c vary with x, then the local value of Δt associated with each grid point will be different from one point to the next. The value of Δt actually employed in Eqs. (7.1) and (7.9) to advance the flow field through the next step in time should be the minimum Δt calculated over all the grid points. [Some CFD applications have employed the 'local time step method', wherein the local values of Δt are used at each grid point in Eqs. (7.1) and (7.9). In this case, the transient variations calculated over many time steps do not hold physically; a type of 'time-warped' flow field is developed, where all the new flow variables calculated for a subsequent time step actually pertain to different total values of time. This 'local time step method' frequently results in a faster convergence to the steady state, that is, fewer total time steps are required to obtain the steady state. On the other hand, the calculated transients have no physical meaning, and some CFD experts wonder openly about the overall accuracy of such a method, even for the final steady state results.] Finally, we note that for a two or three-dimensional flow, an extension of Eq. (7.14) is:

$$\Delta t = \mathrm{Min}(\Delta t_x, \Delta t_y) \tag{7.15a}$$

where

$$\Delta t_x = C\frac{\Delta x}{u+c} \tag{7.15b}$$

and

$$\Delta t_y = C\frac{\Delta y}{v+c} \tag{7.15c}$$

7.5 Selected Applications of the Explicit Time-Dependent Technique

The purpose of this section is to illustrate various applications of the explicit, timedependent technique described in the previous sections of this chapter. These applications contain many of the CFD features that have been discussed throughout these notes.

References [5,6,8] represent the first application of the time-dependent technique to vibrational and chemical non-equilibrium nozzle flows. A purely steady flow analysis of such flows, which involves forward marching from the reservoir to the exit of the nozzle, encounters a saddle-point singularity at the nozzle throat. This singularity greatly complicates steady-state numerical solutions of the flow. On the other hand, as first demonstrated in Refs. [5,6], the time-dependent numerical solution circumvents such problems in the throat region, and therefore constitutes a relatively straightforward numerical solution of such The analysis of vibrational non-.flows equilibrium nozzle flows requires the inclusion of a vibrational rate equation, such as

$$\frac{\partial e_{vib}}{\partial t} = \frac{1}{\tau}[(e_{vib})_{eq} - e_{vib}] - u\frac{\partial e_{vib}}{\partial x} \quad (7.16)$$

where evib is the local non-equilibrium value of molecular vibrational energy per unit mass of gas, (evib)eq is the local equilibrium value, and τ is the vibrational relaxation time which is a function of local p and T. The analysis of chemical nonequilibrium nozzle flows requires the inclusion of species continuity equations—one for each chemical species present in the gas—which are of the form

$$\frac{\partial \eta_i}{\partial t} = \dot{w}_i - u\frac{\partial \eta_i}{\partial x} \quad (7.17)$$

where η_i is the mole–mass ratio (moles of species i per unit mass of mixture), and w_i is the rate of formation (or extinction of species i) due to finite-rate chemical reactions. The form of w_i involves chemical rate constants and the local concentrations of the chemical species. For an introductory development of Eqs. (7.16) and (7.17), see Chaps. 13 and 14 of Ref. [3]. Note that, in the same vein as Eqs. (7.2), (7.3) and (7.4), Eqs. (7.16) and (7.17) are written in the form of a time derivative on the left-hand side, and spatial derivatives on the right-hand side.

In turn, the nonequilibrium variables evib and ηi are calculated in steps of time in the same fashion as ϱ, u and e from Eqs. (7.2), (7.3) and (7.4). Indeed, for the time-dependent solution of non-equilibrium nozzle flows, Eqs (7.2), (7.3) (7.4), (7.16) and (7.17) are coupled, and are solved in the same coupled fashion at each time step as described in Sects. 7.2 and 7.3. However, there is one additional stability restriction brought about by the non-equilibrium phenomena. For explicit solutions of non equilibrium flows, in addition to the CFL criterion discussed in Sect. 7.4, the value of Δt must also be less than the characteristic time for the fastest finite rate taking place in the system. That is

$$\Delta t < B\Gamma$$

where Γ = τ for vibrational non-equilibrium, and $\Gamma = (\partial w_i/\partial \eta_i)^{-1}$ which is an effective chemical relaxation time. (See Refs. [5, 6] for more details.) For this problem, no grid transformation is necessary; the physical and computational planes are one-in-the-same.

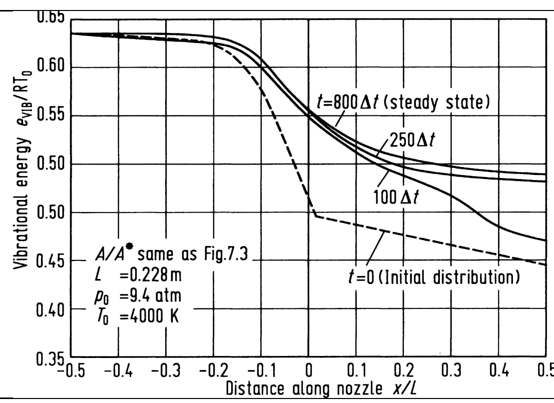
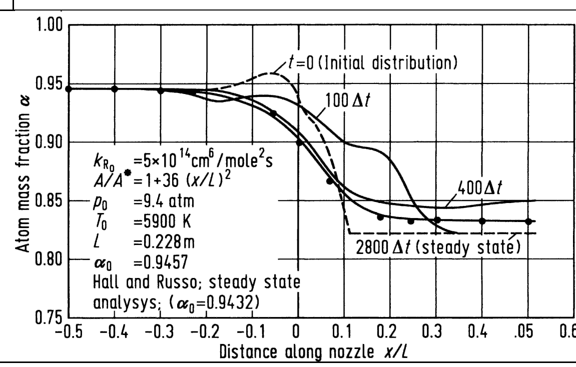


Fig. 7.4 Transient and final steady-state evib distributions for the non-equilibrium expansion of N2 obtained from the present time-dependent analysis

Typical results obtained with the Lax–Wendroff time-dependent technique are shown in Figs. 7.4 and 7.5, from Ref. [5]. The case of the vibrational non-equilibrium expansion of pure N2 is illustrated in Fig. 7.4. Here, the time-dependent nature of the non-equilibrium value of evib as a function of distance through the nozzle is shown. The dashed line represents the assumed initial distribution at t = 0. Several intermediate distributions, after 100 and 250 time steps, are shown, along with the final steady state after 800 time steps. A different case, namely that of the nonequilibrium chemically reacting expansion of dissociated oxygen, is illustrated in Fig. 7.5. Here, the dashed line represents the initially assumed variation of the mass fraction of atomic oxygen through the nozzle at t = 0. Several intermediate curves after 100 and 400 time steps are shown, along with the final, converged steady state after 2800 time steps. This final steady state distribution agrees well with an earlier steady flow solution carried out by Hall and Russo [9], which is shown as the solid circles in Fig. 7.5.

Fig. 7.5 Transient and final steady-state atom mass fraction distributions for the non-equilibrium expansion of dissociating oxygen obtained from the present time-dependent method; the steadystate distribution is compared with the steady-flow analysis of Ref. [9]



Flow Field Over a Supersonic Blunt Body 7.5.2

Here we return to the supersonic blunt body problem discussed in Sect. 1.1. We assume inviscid flow, hence the governing flow equations are represented by Eq. (2.65) with U, F, G, and H given by the inviscid expressions in Sect. 2.9. For the present case, body forces are negligible and hence J = 0.The physical plane is shown at the top of Fig. 7.6; the curve BC is the body and curve AD is the shock wave. The x-coordinates of the shock and body are given by s and b respectively. The local shock detachment distance is given by δ = s–b. During the time-dependent solution, the body is stationary, hence b = b(y). However, the shock wave will change shape and location with time, hence s = s(y, t). Therefore,

$$\delta(y,t) = s(y,t) - b(y) \qquad (7.18)$$

The computational plane (ξ, η) is shown in Fig. 7.6b, and is obtained from the transformation

$$\xi = \frac{x-b}{\delta}; \qquad \eta = y; \ \tau = t \qquad (7.19)$$

where δ is obtained from Eq. (7.18). Note that this transformation is an example of a boundary-fitted coordinate system as discussed in Sect. 5.5.Typical results, obtained from Ref. [10], are shown in Figs. 7.7, 7.8 and 7.9. These results were obtained using the Lax–Wendroff method. In Fig. 7.7, the timedependent wave motion is illustrated, starting from its initially assumed value of t = 0, and progressing to its steady state shape and location after 500 time steps. The time variations of the centreline wave velocity and the stagnation point pressure are shown in Figs. 7.8 and 7.9 respectively. Note in all three Figs. 7.7, 7.8 and 7.9, that the most rapid changes occur at early times, and the steady state is approached rather asymptotically at large times.

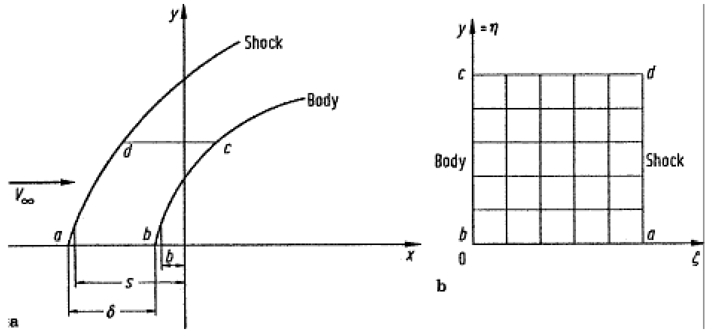


Fig. 7.6 Coordinate system for the blunt body problem

Fig. 7.7 Time-dependent shock wave motion, parabolic cylinder, $M_\infty = 4$

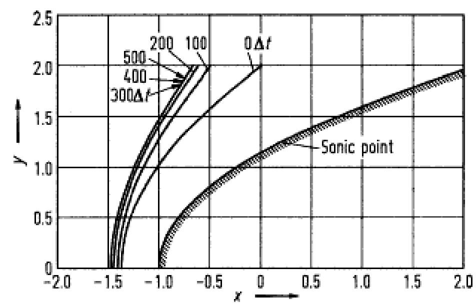


Fig. 7.8 Time variation of wave velocity; parabolic cylinder, $M_\infty = 4$

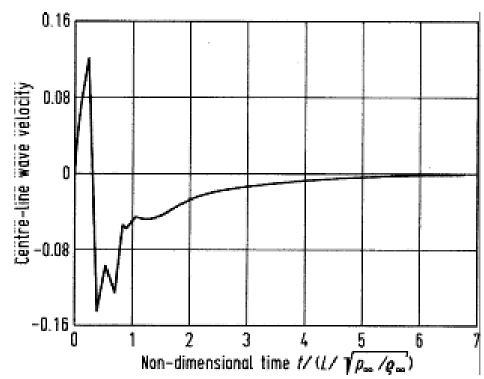
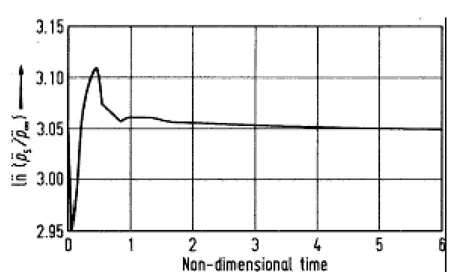


Fig. 7.9 Time variation of stagnation point pressure; parabolic cylinder, $M_\infty = 4$



Consider the flow inside an internal combustion engine as modelled by the pistoncylinder geometry shown in Fig. 7.10. The piston moves up and down inside the cylinder, and the flow enters through the intake valve and exits through the exhaustvalve. The flow field in this problem is truly unsteady, and the objective is to calculate this unsteady flow by means of the time-dependent technique. Here,no asymptotic steady state is ever obtained; rather, a repeatable cyclic flow field is calculated over the complete four-stroke cycle of intake, compression, power and exhaust.We will consider inviscid flow, and hence the governing equations are Eq. (2.65) and the U, F, G, and H column vectors from Sect. 2.9 for an inviscid flow.A boundary-fitted coordinate system is used, where the transformation is

$\xi = x/H(t); \eta - y, \tau = t$

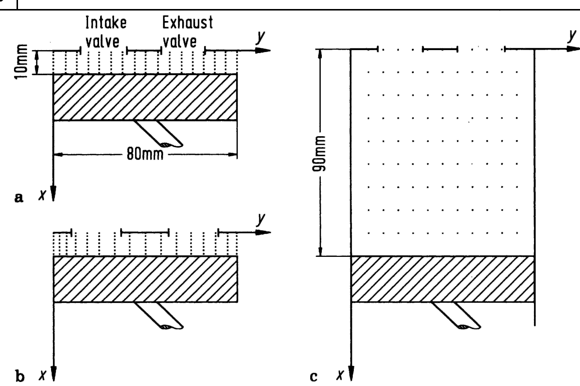


Fig. 7.10 Geometry of two-dimensional cylinder-piston I.C. engine model showing grid arrangement.(a) Piston positioned at TDC, 10 × 17 uniformly spaced grid points; (b) Piston positioned at TDC, 10 × 17 variably spaced grid points (only in y-direction); (c) Piston positioned at BDC, 10×17 uniformly spaced grid points

and where H(t) is the time-varying distance between the top of the cylinder and the top of the piston. Note in Fig. 7.10 that the x coordinate is along the vertical axis of the cylinder, and the y-coordinate is in the radial direction across the cylinder.Results for this flow are shown in Figs. 7.11, 7.12, 7.13 and 7.14, taken from Ref. [11]. The solution is carried out using MacCormack's technique as described in Sect. 7.3. Figures 7.11, 7.12, 7.13 and 7.14 show the flow field associated with bottom dead centre of the intake stroke, three locations of the piston during the compression stroke, near bottom dead centre of the power

stroke, and an intermediate location of the exhaust stroke, respectively. Note that a circulatory flow is created during the intake stroke, and that this circulatory flow persists throughout the fourstroke cycle.	

7.5.4 Supersonic Viscous Flow Over a Rearward-Facing StepWith Hydrogen Injection

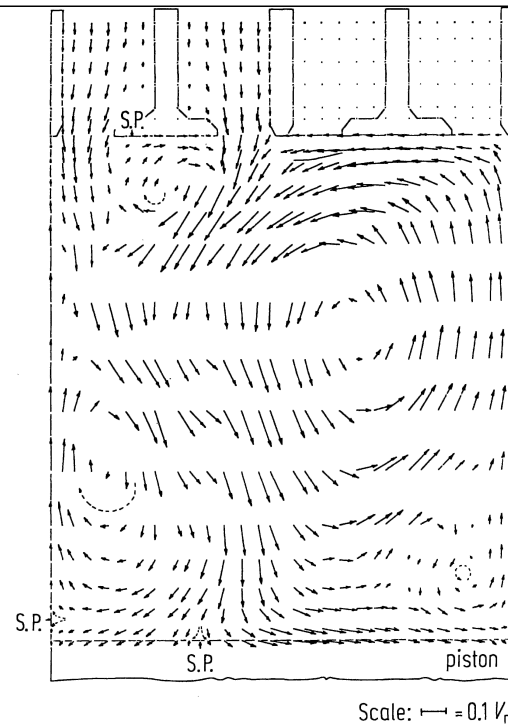
Consider the two-dimensional supersonic viscous flow over a rearward facing step, where H2 is injected into the flow downstream of the step as sketched in Fig. 7.15. Unlike the examples mentioned above, this case deals with the solution of the complete Navier–Stokes Equations, given by Eq. (2.65) with the U, F and G column vectors given in essence in Sect. 2.9 for viscous flow. This system is slightly modified for the presence of mass diffusion, which adds a diffusion term in the energy equation, and adds another equation, namely, the species continuity equation with diffusion terms. (See Refs. [12, 13] for more details.) The numerical technique used here is MacCormack's method discussed in Sect. 7.3. The present calculations were made on a uniform grid throughout the physical space. In combination with the rectangular geometry already existing in the physical plane (as can be seen by examining Fig. 7.15), this means that no grid transformation is needed. Typical results obtained from Refs. [12, 13] are given in Figs. 7.16, 7.17, 7.18 and 7.19. In Fig. 7.16, a velocity vector diagram is shown for the case with no H2 injection. The external Mach number is 2.19, and the Reynolds number	

based on step height is 70,000. These calculations also include a turbulence model patterned after that of Baldwin and Lomax [14].

Note the recirculating separated flow just downstream of the step. Figure 7.17 is a velocity vector diagram with H2 injection. Recirculating separated flows are now seen between the step and the H2 jet, as well as downstream of the jet. Figure 7.18 shows aMach number contour plot of the flow (lines of constant Mach number). Figure 7.19 illustrates the contours of constant H2 mass fraction; this figure serves to define the extent and shape of the jet flow.

Fig. 7.11 Velocity pattern on the intake stroke.

X∗p = 8.78,CA = 161∘,t = 8.95msec = 3080Δt,22×30mesh



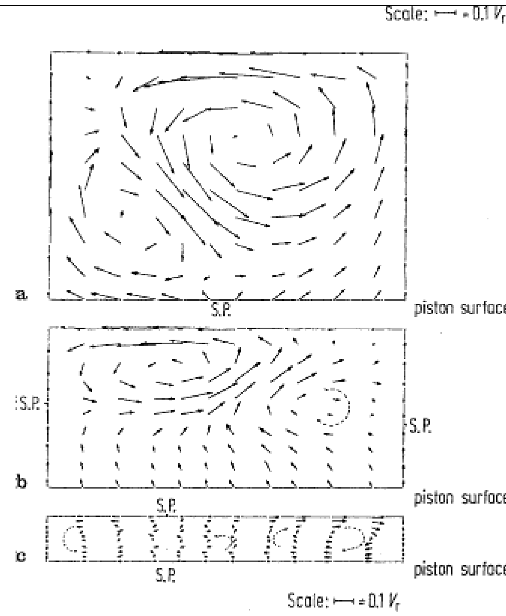


Fig. 7.12 Velocity distributions on compression stroke for the manifold-valve-engine model, 12 × 12 mesh. (**a**) $X_p^* = 5.63$, CA = 261°, t = 14.5 msec = 3970 Δt; (**b**) $X_p^* = 3.56$, CA = 291°, t = 16.2 msec = 4250 Δt; (**c**) $X_p^* = 1.0$, CA = 359°, t = 19.9 msec = 6300 Δt

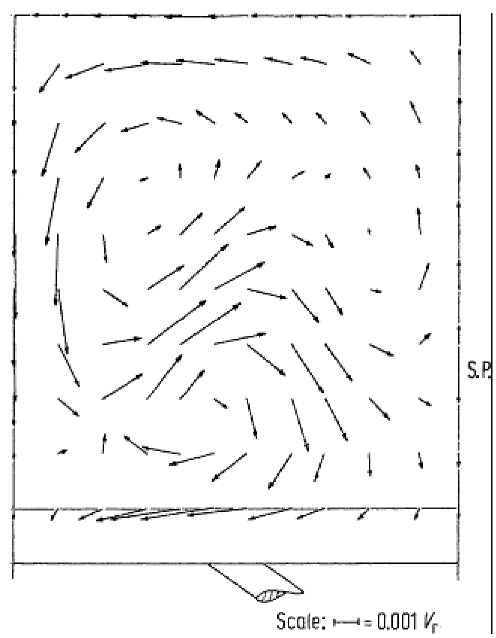


Fig. 7.13 Velocity pattern near end of power stroke; $X_p^* = 8.99$, CA = 539°, t = 29.9 msec = 9950 Δt

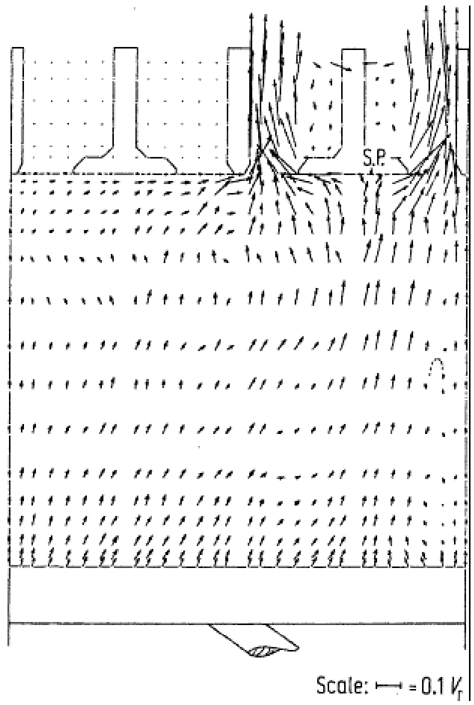


Fig. 7.14 Velocity distribution on exhaust stroke; $X_p^* = 6.99$, CA = 600°, $t = 33.3$ msec = 11560 Δt, 30 × 22 mesh

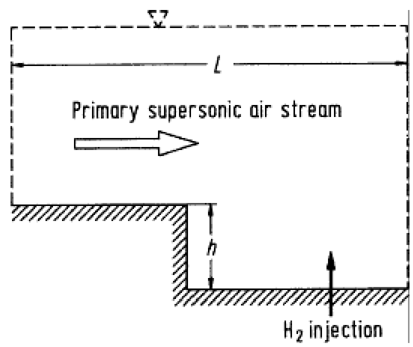


Fig. 7.15 Rearward facing step geometry

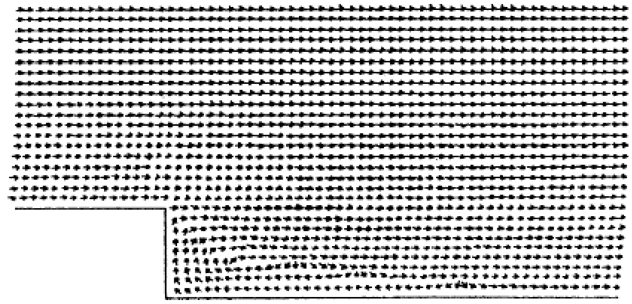


Fig. 7.16 Velocity vectors with no H_2 injection

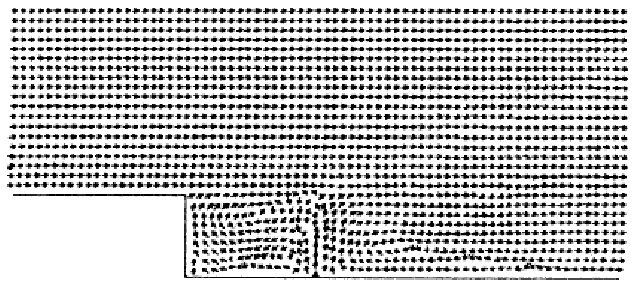


Fig. 7.17 Velocity vectors with H_2 injection

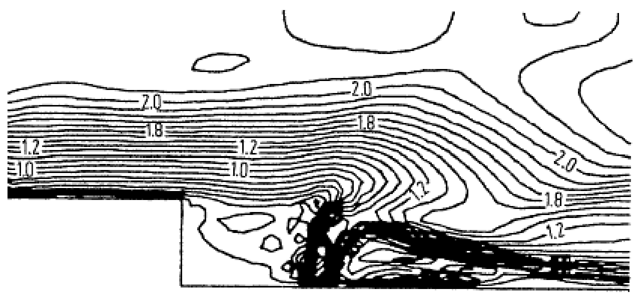


Fig. 7.18 Lines of constant Mach number with H_2 injection

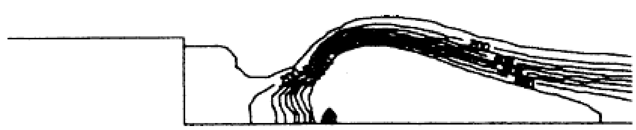


Fig. 7.19 Lines of constant H_2 mass fraction

Supersonic Viscous Flow Over a Base 7.5.5

In a somewhat related fashion, consider the supersonic viscous flow over a base, as illustrated in Fig. 7.20. Here, the same viscous flow equations are used as discussed in Sect. 7.5.4 above. However, for this calculation a stretched grid is used, as given in detail in Sect. 6.4, and as shown in Fig. 6.4. Again, MacCormack's technique is used. Some sample results from Refs. [15,16] are given in Figs. 7.21 and 7.22, which deal with no secondary mass injection at the base. Figure 7.21 shows the velocity vector diagram for the case with an external Mach number of 2.25 and a Reynolds number of 477 000 based on the height of the base. Note the recirculating separated flow downstream of the base. Figure 7.22 illustrates the contours of constant pressure in the flow; the expansion wave around the corner and the recompression shock downstream of the base are clearly seen. Figures 7.23 and 7.24 show the same type of results, except now for the case of air injection from the centre of the base. Note that injection greatly changes the flow field, as can be seen in comparison with Figs. 7.21 and 7.22.

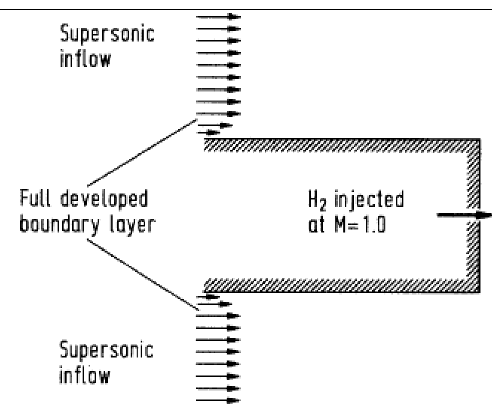


Fig. 7.20 Base flow with mass injection

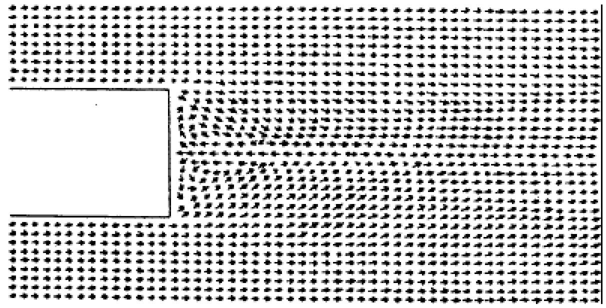


Fig. 7.21 Velocity vectors with no base injection

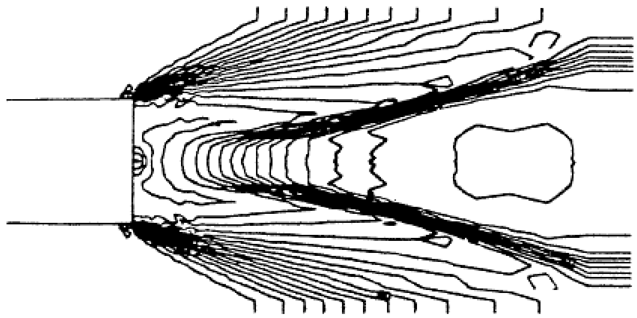


Fig. 7.22 Lines of constant pressure with no base injection

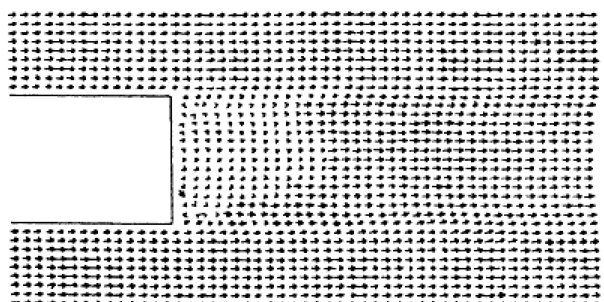


Fig. 7.23 Velocity vectors with injection from the center of the base

Consider the subsonic compressible, viscous two-dimensional flow over an airfoil. The governing equations are the Navier–Stokes equations discussed in Chap. 2. For this application, the choice is made to use the non-conservation form of the equations, namely, Eqs. 2.36(a, b and c), because no shock waves will be present in theflow. MacCormack's method is used. Consider the airfoil and the elliptically generated boundary-fitted grid shown in Figs. 6.8 and 6.9, as discussed in Sect. 6.5,and as taken from Refs. [17, 18]. Calculated results for a free stream Mach number of 0.5 and a Reynolds number based on chord length of 100 000 (this is a low Reynolds number flow, which was the objective of the study in Ref. [18]) are shown in Figs. 7.25, 7.26 and 7.27. The angle-of-attack in these figures is zero. These figures illustrate the instantaneous flow over a Wortmann airfoil at different times. In Figs. 7.25 and 7.26, the flow is laminar, and it separates over the top surface of the airfoil at about the maximum thickness point. The flow is clearly unsteady, as can beseen by comparing Fig. 7.25(a, b and c); there is a rather periodic flow fluctuation over the rearward portion of the airfoil, as well as downstream of the trailing edge.The calculation of such unsteady flows, especially in situations where they may be unexpected, is one of the major advantages of the time-dependent method in comparison to steady-state analyses. In Fig. 7.27, the flow is treated as turbulent; note that in this case the flow is attached.	

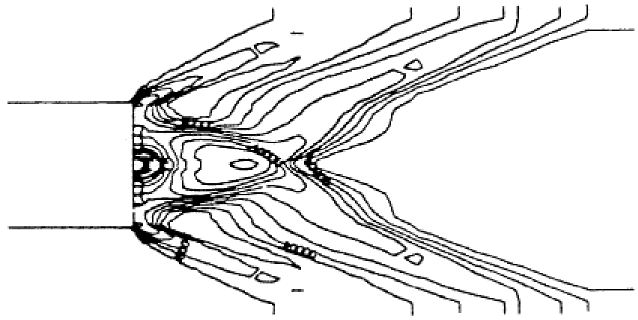


Fig. 7.24 Lines of constant pressure with injection from the center of the base

This author has many more examples of CFD applications from the work of his graduate students; those listed in Sect. 7.5 are but a small fraction. They are picked for discussion in these notes on a rather arbitrary basis. Time and space do not allow further listing and discussion. Also, this brings to an end our introduction to CFD. It is the author's hope that these notes have been a reasonable beginning for the unitiated reader, and that he or she can now greatly expand his or her horizons by reading the more advanced literature on CFD. If such advanced reading is indeed more easy after studying the present notes, then this author has accomplished his goal In recent years, some modern texts on CFD have been published (Refs. [19–23]); these texts are recommended for advanced studies of the subject. In particular, Fletcher's two volumes (Refs. [19, 20]) contain a nice theoretical discussion of the subject. Of special note are the two volumes by Hirsch (Refs. [21, 22]); these volumes represent an authoritative presentation of the mathematical and numerical fundamentals of CFD, the modern techniques used in CFD, and how these techniques are used in various practical applications. Reference [23], by Hoffmann, is a crisp presentation of CFD for use by engineers. All of these books are recommended for more advanced study of computational fluid dynamics. Also, for an extended presentation of the elementary, introductory ideas contained in the present book, as well as a lengthy discussion of the overall philosophy of CFD and its role in modern engineering, see the book by the present author (Ref. [24]); this is written for a senior-level undergraduate course in CFD, and assumes absolutely no prior knowledge of the subject. This author wishes you happy reading, and happy computing in your further expeditions into the world of computational fluid dynamics.

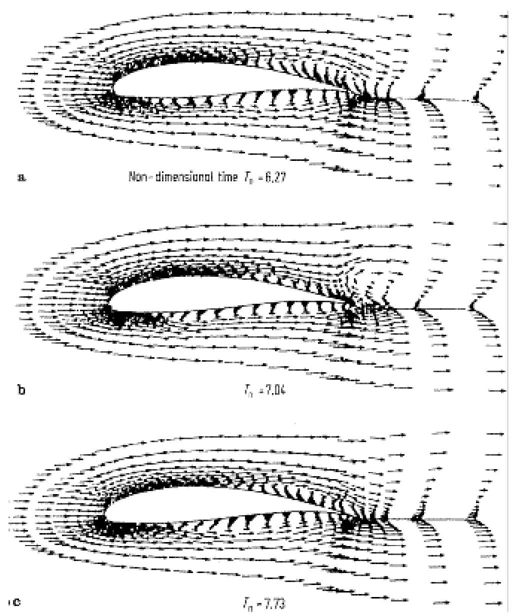


Fig. 7.25 Velocity vector diagrams at three different non-dimensional times for purely laminar flow (Re = 1000 000, M = 0.5, Alpha = 0.0 deg.). **(a)** Non-dimensional time T_n = 6.27. **(b)** T_n = 7.04.

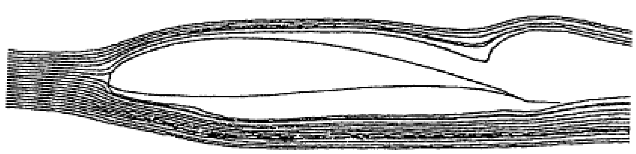


Fig. 7.26 Instantaneous streamlines over Wortmann airfoil (FX63-137)—laminar flow (unsteady results) (Re = 100000, M = 0.5, Alpha = 0.0 deg.) Non-dimensional time T_n = 7.04

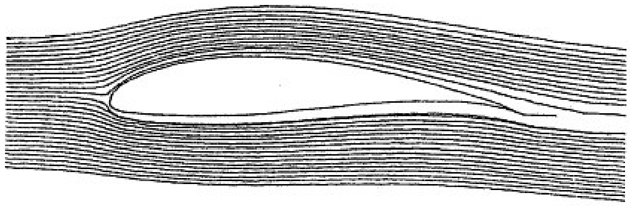


Fig. 7.27 Streamlines over Wortmann airfoil (FX63-137)—turbulent flow (Re = 100 000, M = 0.5, Alpha = 0.0 deg.)

1. Anderson, John D., Jr., Fundamentals of Aerodynamics, 2nd Edition McGraw-Hill, New York, 1991.
2. Anderson, John D., Jr., 'Computational Fluid Dynamics—An Engineering Tool?' in A.A. Pouring (ed.), Numerical Laboratory Computer Methods in Fluid Dynamics, ASME, New York, 1976, pp. 1–12.
3. Anderson, J.D., Jr., Modern Compressible Flow: With Historical Perspective, 2nd Edition McGraw-Hill, New York, 1990.
4. Ames Research Staff, 'Equations, Tables, and Charts for Compressible Flow,' NACA Report 1135, 1953.
5. Anderson, J.D. Jr., 'A Time-Dependent Analysis for Quasi-One-Dimensional Nozzle Flows with Vibrational and Chemical Nonequilibrium,' NOLTR 69-52, Naval Ordnance Laboratory, White Oak, MD, 1969.
6. Anderson, J.D., Jr., 'A Time-Dependent Analysis for Vibrational and Chemical Nonequilibrium Nozzle Flows,' AIAA Journal, Vol. 8, No. 3, March 1970, pp. 545–550.
7. MacCormack, R.W., 'The Effect of Viscosity in Hypervelocity Impact Cratering,' AIAA Paper No. 69–354, 1969.
8. Anderson, J.D., Jr., 'Time-Dependent Solutions of Nonequilibrium Nozzle Flow—A Sequel,' AIAA Journal, Vol. 5, No. 12, Dec. 1970. pp. 2280–2282.
9. Hall, J.G. and Russo, A.L., 'Studies of Chemical Nonequilibrium in Hypersonic Nozzle Flows,' AFOSR TN 59-1090, Cornell Aeronautical Laboratory Report AD-1118-A-6, November 1969.
10. Anderson, J.D., Jr., 'On Hypersonic Blunt Body Flow Fields Obtained with a Time-Dependent Technique,' NOLTR 68–129, Naval Ordnance Laboratory, White Oak, MD, August 1968.
11. Dallospedale, C.L., 'A Numerical Solution for the Two-Dimensional Flowfield in an Internal Combustion Engine with Realistic Valve-Geometry,' M.S. Thesis, Department of Aerospace Engineering, University of Maryland, College Park, MD, 1978.

9 مدخل الي طريقة العناصر المنتهية (FEM) في ديناميكيات الموائع الحسابية (CFD)[9]

9.1 مدخل

The finite element method (FEM) is a numerical technique for solving partial differential equations (PDE's).	طريقة العناصر المنتهية (Finite element method) أو يطلق عليها أيضاً تحليل العناصر المنتهية هي طريقة تحليل عددي لإيجاد الحلول التقريبية للمعادلات التفاضلية الجزئية بالإضافة إلى الحلول التكاملية. يعتمد الحل إما على إلغاء المعادلات التفاضلية الجزئية نهائياً (في الحالات الساكنة) أو تقريب المعادلات التفاضلية الجزئية إلى معادلات تفاضلية نظامية والتي يكون من الممكن

9 http://ar.wikipedia.org/wiki/%D8%B7%D8%B1%D9%8A%D9%82%D8%A9_%D8%A7%D9%84%D8%B9%D9%86%D8%A7%D8%B5%D8%B1_%D8%A7%D9%84%D9%85%D9%86%D8%AA%D9%87%D9%8A%D8%A9#.D8.AA.D8.B7.D8.A8.D9.8A.D9.82.D8.A7.D8.AA and [Wendt 2009], Ch. 10.

حلها باستخدام عدة طرق كطريقة أويلر (Euler) أو رونغي–كوتا (Runge-Kutta).

Its first essential characteristic is that the continuum field, or domain, is subdivided into cells, called elements, which form a grid. The elements (in 2D) have a triangular of a quadrilateral form and can be rectilinear or curved. The grid itself need not be structured. With unstructured grids and curved cells, complex geometries can be handled with ease.

The second essential characteristic of the FEM is that the solution of the discrete problem is assumed a priori to have a prescribed form. The solution has to belong to a function space, which is built by varying function values in a given way, for instance linearly or quadratically between values in nodal points. The nodal points, or nodes, are typical points of the elements such as vertices, mid-side points, mid-element points, etc. Due to this choice, the representation of the solution is strongly linked to the geometric representation of the domain.

The third essential characteristic is that a FEM does not look for the solution of the PDE itself, but looks for a solution of an integral form of the PDE. The most general integral form is obtained from a *weighted residual formulation*. By this formulation the method acquires the ability to naturally incorporate differential type boundary conditions and allows easily the construction of higher order accurate methods. The ease in obtaining higher order accuracy and the ease of implementation of boundary conditions form a second important advantage of the FEM.

A final essential characteristic of the FEM is the modular way in which the discretization is obtained. The discrete equations are constructed from contributions on the element level which afterwards are *assembled*.

9.2 شرح طريقة العناصر المنتهية

سوف نستخدم مثالين بسيطين لشرح طريقة العناصر المنتهية، والتي من خلالها من الممكن استخلاص الطريقة العامة. في النقاش التالي، يجب على القارئ أن يكون متفهما لمبادئ علم الحسبان والجبر الخطي.

P1 هي مسألة **أحادية البعد**، معطاة على الشكل التالي:

$$\text{P1}\ :\begin{cases} u''=f \text{ in } (0,1), \\ u(0)=u(1)=0, \end{cases}$$

حيث f معلوم و u هو تابع مجهول للمتحول x ، و u'' هو المشتق الثاني للتابع u بالنسبة للمتحول x.

المسألة ثنائية البعد البسيطة هي مسألة ديركلت (Dirichlet) وتعطى على الشكل التالي:

$$\text{P2}\ :\begin{cases} u_{xx}+u_{yy}=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

حيث Ω هي منطقة مفتوحة متصلة في المستوي الثنائي البعد (x,y) الذي تكون حدوده $\partial\Omega$ هي عبارة عن مضلع ذو شكل جميل. و u_{xx} و u_{yy} هي المشتقات الثانية للمتحولين x و y على الترتيب.

من الممكن حل المسألة أحادية البعد بحساب المشتق العكسي .لكن هذه الطريقة في حل مسألة القيمة الحدية (boundary value problem) تصلح لحل المسائل أحادية البعد ولا يمكن تعميمها إلى مسائل ذات أبعاد أعلى أو مثال لها الشكل $u+u''=f$ ولهذا السبب كان من الضروري تطوير طريقة العناصر المنتهية، بدءاً من البعد الأحادي وتعميمها على الأبعاد الأعلى.

الشرح هنا سوف يتم على مرحلتين والتي تعكس المرحلتين الأساسيتين الواجب تطبيقهما لحل مسألة القيمة الحدية باستخدام طريقة العناصر المنتهية:

الخطوة الأولى: تبسيط مسألة القيمة الحدية (boundary value problem) إلى شكل بسيط تنتفي معه الحاجة إلى استخدام الحاسب للحل، بل يكون من الممكن حلها يدوياً باستخدام الورقة والقلم.

الخطوة الثانية: هي التقطيع، حيث يتم تجزئة الشكل إلى عناصر منتهية وحل كل عنصر على حدة.

بعد هذه الخطوة سيكون لدينا صيغة متكاملة لحل مسائل ذات درجات عالية لكن يجب أن تكون خطية والتي حلولها ستكون حلاً تقريبياً لمسألة القيمة الحدية. ومن ثم يتم برمجة هذه الطريقة على الحاسوب.

9.3 الصيغة المتحولية (variational formulation)

Variational formulation = The minimization of an energy integral over the domain.

الصيغة المتحولية هي صيغة طبيعية تكاملية لطريقة العناصر المنتهية (FEM) و لكن في ميدان الميكانيك الموائع – بشكل عام – هو غير ممكن ان توضع الصيغة المتحولية (variational formulation).

الخطوة الأولى هو تحويل P1 و P2 إلى مكافئاتها المتحولية .إذا كان u هو حل لـــP1 ، عندها من أجل أي دالة متصلة v يحقق شروط الانتقال الحدي، مثلاً $v = 0$:عند $x = 0$ و $x = 1$، يكون لدينا

(1)

وبشكل معاكس، من أجل قيمة معطاة لـــ u فإن (1) تكون محققة من أجل أي دالة متصلة $v(x)$ وعندها من الممكن أن يبرهن أن u ستكون حلاً لـــ) P1 برهان هذا ليس بالأمر السهل وهو يعتمد على فضاء سوبوليف.)

وباستخدام التكامل بالأجزاء على يمين المعادلة (1) سنحصل على مايلي:

(2)

حيث تم افتراض أن.$v(0) = v(1) = 0$

9.3.1 برهان يظهر وجود حل وحيد

من الممكن اعتبار أن $H_0^1(0,1)$ هو عبارة عن تابع مستمر مطلق للثنائية (0,1) بحيث أن 0 عند x = 0 و) x = 1انظر فضاء سوبوليف .). مثل هذه التوابع تكون ضعيفة (قابلة للاشتقاق مرة واحدة) وتكشف عن الخريطة الخطية الثنائية المتناظرة ϕ ومن ثم تعرف جداء داخلي الذي يحول $H_0^1(0,1)$ إلى فضاء هلبرت .ومن ناحية أخرى، فإن الطرف الأيسر $\int_0^1 f(x)v(x)dx$ هو أيضاً جداء داخلي، ولكن هذه المرة على الفضاءLp $L^2(0,1)$.وتطبيق لمبرهنة تمثيل رايسز على فضاءات هلبرت يظهر أنه يوجد حل وحيد u يحل (2) وبالتالي يحل المسألة.P1

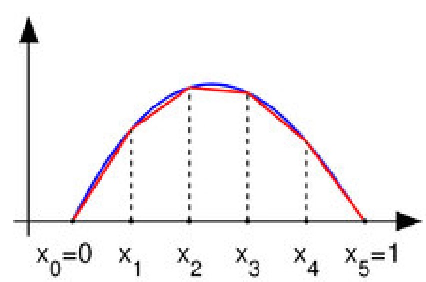
9.3.2 الصيغة المتحولية لـــP2

إذا تم التكامل بالأجزاء باستخدام مبرهنة غرين حيث نجد أنه إذا كان u هو حل لـــ P2 ، فإنه من أجل أي v يكون

$$\int_\Omega fv\,ds = -\int_\Omega \nabla u \cdot \nabla v\,ds = -\phi(u,v),$$

حيث ∇ تحقق التدرج وترمز إلى الجداء الداخلي في المستوي ثنائي البعد.

9.4 التقطيع (Discretization)



التابع H^1_0 مع القيم الصفرية عند نقاط النهاية (زرقاء)، والتقريب الخطي الجزئي للمنحني (حمراء).

الفكرة الأساسية في طريقة العناصر المنتهية هو استبدال المسألة الخطية ذات الأبعاد اللانهائية: أوجد قيمة $u \in H_0^1$ بحيث أن

$$\forall v \in H_0^1,\ -\phi(u,v) = \int fv$$

بصيغة بعدية منتهية:

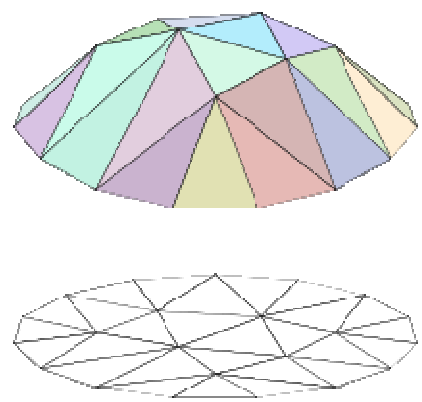
(3)أوجد $u \in V$ such that

$$\forall v \in V,\ -\phi(u,v) = \int fv$$

حيث V هو فضاء جزئي خطي ذو عدد أبعاد منته من H_0^1. هناك العديد من الخيارات لـــ V. لكن في طريقة العناصر المنتهية نعتبر V على أنها فضاء للأجزاء الخطية للتابع.

في المسألة P1 ، نأخذ المقطع (0,1) باختيار x n قيم من $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ ونعرف V على الشكل:

$$V = \{v : [0,1] \rightarrow \mathbb{R}\ : v \text{ is continuous}, v|_{[x_k, x_{k+1}]} \text{ is linear for } k = 0, ..., n, \text{ and } v(0) = v(1) = 0\}$$



حيث نعرف $x_0 = 0$ و $x_{n+1} = 1$. لاحظ أن التوابع في V هي توابع غير قابلة للاشتقاق بالاعتماد على التعريف المبدئي للحسبان. إذا كان $v \in V$ فإن المشتق يكون عادة غير معرف عند أي $x = x_k,\ k = 1,\ldots,n$. لكن يوجد مشتق عند كل قيمة للمتحول x ومن الممكن استخدام هذا المشتق لغرض التكامل بالأجزاء.

تابع خطي مقطع في المستوي ثنائي الأبعاد.

من أجل المسألة P2 نحتاج أن تكون V عبارة عن مجموعة من التوابع من Ω. في الشكل الموضح على اليسار، يظهر تثليث مضلعي لمنطقة مضلعية من 15 ضلع Ω في المستوي (في الأسفل)، والتابع الخطي المجزأ (ملوناً، في الأعلى) لهذا المضلع الذي يكون خطياً على كل مثلث من التثليث. حيث أن الفضاء V سيحتوي على توابع تكون خطية على كل مثلث من التثليث المختار.

تظهر V مكتوبة على الشكل V_h في بعض المراجع، وذلك بسبب أنه يوجد هدف في الحصول على حلول أدق وأدق للمسألة المتقطعة (3) الذي سيكون إلى حد ما سيؤدي إلى حد المسألة الأصلية في إيجاد القيم الحدية للمسألة P2.يتم عنونة التثليث باستخدام معامل ذو قيمة حقيقية $h > 0$ والذي يكون ذو قيمة صغيرة. سوف يتم ربط هذا المعامل بحجم أكبر مثلث وسطي الحجم في التثليث. وعندما نزيد تجزئة التثليث فإن فضاء التقطيع الخطي V يجب أن يتغير مع h كما يوضح الترميز.V_h

9.5 الصيغة القوية والصيغة الضعيفة احد المسائل القيمة الحدية (boundary value problem)

10 تمارين

Writing down the governing equations onto the paper
developing the appropriate numerical solution of these equations
writing the C++ / FORTRAN program and putting it into the computer
going through all the trials and tribulations of making the program work properly

مدخل الى الحرق الحسابي (Introduction to Numerical Combustion)

Based on
Theroretical and Numerical Combustion (Thierry Poinsot, Denis Veynante) and
Introduction to Combustion – Concepts and Applications, 2nd edition (Stephen R. Turns)
و مراجع اخرى

Introduction to mass transfer[10] 11

[10] From [Turns], pp. 83-105

12 معادلات الاستمرارية لسرايين تفاعلية (Conservation equations for reacting flows)

12.1 اشكال عامة (General forms)

12.1.1 اختيار المتحولة البداهية (primitive variables)

Some Important Chemical Mechanisms 13

(The H2-O2 System) [11] 13.1

[11] From [Turns], 148-152

Laminar premixed flames and Laminar Diffusion flames 14

Introduction to Turbulent Flows 16

Turbulent Premixed and Nonpremixed flames 17

Burning of solids 18

19 مراجع

I 19.1

1. [Anderson 1991] Anderson, John D., Jr., Fundamentals of Aerodynamics, 2nd Edition McGraw-Hill, New York, 1991
2. , Springer 8[Ferziger, Peric] J. Ferziger und M. Peric, *Numerische Strömungsmechanik*, 200 Verlag.
3. [Wessling] Pieter Wesseling, *Principles of Computational Fluid Dynamics,* 2000, Springer Verlag.
4. [Wendt 2009] John F. Wendt, *Computational Fluid Dynamics – an Introduction (a von Karman Institute Book),* Third Edition, 2009, Springer Verlag
5. [صديق] محمد هاشم الصديق (الإستاذ المشارك بشعبة هندسة الموائع قسم الهندسة الاميكانيكية / كلية الهندسة والعمارة، جامعة الخرطوم،msiddiq@yahoo.com)، **ميكانيك الموائع**، الاصدارة الثانية، 2006
6.

مجمع اللغة العربية

..

II 19.2

1. [Poinsot, Veynante] Thierry Poinsot, Denis Veynante; *Theroretical and Numerical Combustion*
2. [Turns] Stephen R. Turns; *Introduction to Combustion – Concepts and Applications*, 2nd edition

20 ملحقات (Apprendices)

20.1 ملحق أ: مضمون كتاب "ميكانيك الموائع" لمحمد هاشم الصديق

مضمون [صديق] محمد هاشم الصديق (الإستاذ المشارك بشعبة هندسة الموائع قسم الهندسة الالميكانيكية / كلية الهندسة والعمارة، جامعة الخرطوم،msiddiq@yahoo.com)، **ميكانيك الموائع**، الاصدارة الثانية، 2006 هو التالي:

20.2 ومضمون كتاب [Ferziger, Peric]

مدخل الى التحليل العددي (بالإنجليزية: Numerics)

(بالإنجليزية: Components of a numerical method)

(بالإنجليزية: Mathematical model)

(بالإنجليزية: Discretization method)

(بالإنجليزية: Coordinate and base vector systems)

(بالإنجليزية: Numerical mesh)

(بالإنجليزية: Finite Approximations)

(بالإنجليزية: Solution method)

(بالإنجليزية: Convergence criteria)

اساسيات ديناميك الحرارية (بالإنجليزية: Thermodynamics)

(بالإنجليزية: Finite Difference Methods)

(بالإنجليزية: Finite Volume Methods)

طريقة العناصر المنتهية (FEM)

(بالإنجليزية: Solving linear equation systems)

(بالإنجليزية: Solving the Navier-Stokes Equations)

(بالإنجليزية: Computation Methods for complex flow areas)

(بالإنجليزية: Simulation of turbulence)

(بالإنجليزية: Compressible Fluids)

(بالإنجليزية: Efficiency and accuracy)

(بالإنجليزية: Special Topics)

(بالإنجليزية: Combustion)

20.3 مواضيع اضافية

(بالإنجليزية: CFD Applications in Energy Engineering)

(بالإنجليزية: CFD Applications in Aeronautics)

(بالإنجليزية: CFD Applications in Space Technology

20.4 ملحق أ: مضمون كتاب Theroretical and Numerical Combustion (Thierry Poinsot, Denis Veynante)

مضمون الكتاب هو التالي:

20.5 ملحق ب: مضمون Introduction to Combustion – Concepts and Applications, 2nd edition (Stephen R. Turns)

مضمون الكتاب هو التالي:

Dictionnary

Content

A

English	Deutsch	عربي

B

English	Deutsch	عربي

English	**Deutsch**	**عربي**
calculation	Berechnung	
Continuity equation	Kontinuitätsgleichung	معادلة الاستمرارية
Conservation form		
conservation form		الشكل التحفظي
control volume		حجم التحكم

English	Deutsch	عربي
derivate	Ableitung, Differentialquotient	
differential		تفاضلي
distinct	verschiedenr	

English	Deutsch	عربي
explicit		

finite difference method		
fluid element		عضو مائع
fluid dynamics		حركية الموائع
Flow	Fluss, Stömung	سريان
flow field		
finite-difference methods	Finite-Differenzen Methoden	طرق الفرق المحدود
flux	Strom	سريان
friction	Reibung	احتكاك

G

govering equation		معادلة اساسية
grid		

hyperbolic		

integral		تكاملي
incorporate		
incompressible	inkompressibel	لا انضغاطي
infinitesimal		موحل في الصغر
inviscid	nicht zähflüssig	لا لزجي
irrotational	nicht rotierend	لا دوراني
integral form		

K

L

linear algebra	Linerare Algebra	علم الحساب الجبر الخطي

M

momentum		كمية التحرك

N

numerical analysis		التحليل العددي
normal		عمودية

O

One-dimensional	eindimensional	أحادية البعد

parabolic		
panel	Gruppe, Runde	مؤطَّرة
property	Eigenschaft	خصوصية
partial differential equations		المعادلات التفاضلية الجزئية

Q

R

(chemical) reaction		تفاعل كميائي
rectangular		

shear	Scherung	قص
Shear stress	Scherspannung	الإجهاد القصي
slope	Anstieg (einer Funktion) (math.)	
steady-state		
source	Quelle	نبع
system	System	منظومة
stress	Spannung (Druckvektor)	اجهاد
Substantial Derivate		الاشتقاق الكبير

time-dependend method		
Transient		
tangential		مماسة

Uniform		

Viscous		لزجي
source	Quelle	نبع
variable x		متحول x

W

X

Y

Z

calculation	Berechnung	
incorporate		
time-dependend method		
steady-state		
flow field		
Transient		
hyperbolic		
parabolic		
incompressible	inkompressibel	لا انضغاطي
source	Quelle	نبع
vortex	Wirbel	دوامة مائية
panel	Gruppe, Runde	مؤطَّرة
numerical analysis		التحليل العددي
inviscid	nicht zähflüssig	لا لزجي
finite-difference methods	Finite-Differenzen Methoden	طرق الفرق المحدود
irrotational	nicht rotierend	لا دوراني
property	Eigenschaft	خصوصية
govering equations		المعادلاب الاساسية

integral form		
system		منظومة
control volume		حجم التحكم
normal		عمودية
tangential		مماسة
flux	Strom	سريان
Uniform		
rectangular		
grid		
stress	Spannung (Druckvektor)	اجهاد
shear	Scherung	قص
	Scherspannung	الإجهاد القصي
S		
stress	Spannung σ (hat Einheit N/m^2 , d.h. die gleiche Einheit wie ein Druck)	الاجهاد
Substantial Derivate		الاشتقاق الكبير

V		
Viscous		لزجي
Flow	Fluss, Stömung	سريان
calculation	Berechnung	
incorporate		
time-dependend method		
steady-state		
flow field		
Transient		
hyperbolic		
parabolic		