Ex III $f : \mathbb{R} \longrightarrow \mathbb{R} \,/\, f(x) = \dfrac{1}{1+3\cos^2 x}$

1. $f(x+\pi) = \dfrac{1}{1+3\cos^2(x+\pi)} = \dfrac{1}{1+3\cos^2 x} = f(x) \Rightarrow f$ est π_périodiq

$f(-x) = \dfrac{1}{1+3\cos^2(-x)} = \dfrac{1}{1+3\cos^2 x} = f(x) \Longrightarrow f$ est paire

2. $\displaystyle\int_0^{2\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx$ [car f est π_périodique]

$\displaystyle = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx$ [" " "]

$\displaystyle = 4\int_0^{\frac{\pi}{2}} f(x)\,dx$ [car f est paire].

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+3\cos^2 x} \underset{\substack{t = tg x \\ dt = (1+t^2).dx}}{=\!=} \int_0^{+\infty} \frac{\frac{dt}{1+t^2}}{1+3\frac{1}{1+t^2}} = \int_0^{+\infty} \frac{dt}{1+t^2+3} = \int_0^{+\infty} \frac{dt}{4+t^2}$$

$$= \frac{1}{4}\int_0^{+\infty} \frac{dt}{1+\left(\frac{t}{2}\right)^2} = \frac{1}{4} \cdot 2 \,\mathrm{Arctg}\, \frac{t}{2}\Big]_0^{+\infty} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

Donc $\displaystyle\int_0^{2\pi} f\,dx = 4 \cdot \frac{\pi}{4} = \pi$.

3. $$\int_0^{2\pi} \frac{x\,dx}{1+3\cos^2 x} \underset{\substack{t = 2\pi - x \\ dt = -dx}}{=\!=} \int_{2\pi}^{0} \frac{(2\pi - t)(-dt)}{1+3\cos^2(2\pi - t)} = \int_0^{2\pi} \frac{(2\pi - t)}{1+3\cos^2 t}\,dt$$

$$= 2\pi \int_0^{2\pi} \frac{dt}{1+3\cos^2 t} - \int_0^{2\pi} \frac{x\,dx}{1+9\cos^2 x} \Longrightarrow 2\int_0^{2\pi} \frac{x\,dx}{1+3\cos^2 x} = 2\pi.\pi$$

$$\Longrightarrow \int_0^{2\pi} \frac{x\,dx}{1+3\cos^2 x} = \pi^2.$$

Ex IV 1. $\displaystyle\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{\ln x}{(1+x)^2}\,dx$

$\begin{cases} u = \ln x \Rightarrow du = \dfrac{dx}{x} \\ dv = \dfrac{dx}{(1+x)^2} \Rightarrow v = \dfrac{-1}{1+x} \end{cases}$

$$\int_a^1 \frac{\ln x}{(1+x)^2}\,dx = -\frac{\ln x}{1+x}\Big]_a^1 + \int_a^1 \frac{dx}{x(1+x)}$$

$$= \frac{\ln a}{1+a} + \int_a^1 \left[\frac{1}{x} - \frac{1}{1+x}\right] dx$$

$$= \frac{\ln a}{1+a} + \left[\ln x - \ln(1+x)\right]_a^1 = \frac{\ln a}{1+a} + \left[0 - \ln 2 - \ln a + \ln(1+a)\right]$$